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How to arrange a Singles' Party: Coalition Formation in Matching Game [a]

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Abstract: The study addresses important issues relating to computational aspects of coalition formation. However, finding payoffs—imputations belonging to the core—is, while almost as well known, an overly complex, NP-hard problem, even for modern supercomputers. The issue becomes uncertain because, among other issues, it is unknown whether the core is non-empty. In the proposed cooperative game, under the name of singles, the presence of non-empty collections of outcomes (payoffs) similar to the core (say quasi-core) is fully guaranteed. Quasi-core is defined as a collection of coalitions minimal by inclusion among non-dominant coalitions induced through payoffs similar to super-modular characteristic functions (Shapley, 1971). As claimed, the quasi-core is identified via a version of P-NP problem that utilizes the branch and bound heuristic and the results are visualized by Excel spreadsheet.

Keywords: stability; game theory; coalition formation.

1. Introduction

It is almost a truism that many university and college students abandon schooling soon after starting their studies. While some students opt for incompatible education programs, the composition of students following particular programs may not be optimal; in other words, students and programs are mutually incompatible. Indeed, so-called mutual mismatches of priorities were among the reasons (Võhandu, 2010) behind the unacceptably high percentage of students in Estonian universities and colleges dropping out of schools, wasting their time and the entitlement to government support. However, matching students and education programs more optimally could mitigate this problem.

Similar problems have been thoroughly studied (Roth, 1990; Gale, 1962; Berge, 1958...) leading, perhaps, L. Võhandu to propose a way, in this wide area of research, to solve the problem of students and programs mutual incompatibility by introducing "matching total" as the sum of duplets—priorities selected within two directions—horizontal priorities of students towards programs, and vertical priorities of programs towards students. The best solution found among all possible horizontal and vertical duplet assignments, according to LV, is where the sum reaches its minimum.

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[a] *A thesis of this paper was presented at the Seventh International Conference on Game Theory and Management (GTM2013), June 26-28, 2013, St. Petersburg, Russia.*

Finding the best solution, however, is a difficult task. Instead, LV proposed a greedy type workaround. In the author's words, the best solution to the problem of matching students and programs will be close enough (consult with Carmen et al., 2001) to a sum of duplets accumulated while moving along ordering of duplets in non-decreasing direction. It seems that LV's proposal to the solution is a typical approach in the spirit of classical utilitarism, when the sum of utilities has to be maximized or minimized (Bentham, The Principals of Morals and Legislation, 1789; Sidgwick, The Methods of Ethics, London 1907).

As noted by Rawls in "Theory of Justice", the main weakness of utilitarian approach is that, when the total **max** or **min** has been reached, those members of society at the very low utility levels will still be receiving very low compensations for incapacity, such as transfer payments to the poor. Arguing for the principal of "***maxima of the lowest***", referred to as the "Second Principal of Justice", Rawls suggested an alternative to the utilitarian approach. The motive driving this study is similar. We address by example an alternative to conventional core solution in cooperative games, along the lines of monotonic game (Mullat, 1979), whereby the lowest incentive/compensation should be maximized. The reader studying matching problems can also find useful information about these issues, where a number of ways of constructing an optimal matching strategy have been discussed (Veskioja, 2005).

Learning by example is of high value because the conventional core solution in cooperative games cannot be clearly explained unless the readers are sufficiently familiar with utopian reality—a reality that sometimes does not exist. Thus, a rigorous set up of a simple game will be presented here, aiming to explain the otherwise rather complicated intersection of interests. More specifically, we hope to shed light on what we call a Singles-Game. It should be emphasized that, even though the game primitives represent an independent mathematical object in a completely different context, we have still "borrowed" the idea of LV duplets to estimate the benefits of matching. For this reason, we changed the nomenclature of duplets to mutual risks in order to justify the scale of payoffs—the incentives and compensations.

The rest of the paper is organized as follows. We start with the preliminaries, where the game primitives are explained. In Section 3, we introduce the core concept of conventional stability in relation to the Singles-game. In Section 4, the reader will come across an unconventional theory of kernel coalitions, and nuclei coalitions, minimal by inclusion in accordance with the formal scheme. In Section 5, we continue explaining our techniques and procedures used to locate stable outcomes of the game. The study ends with conclusions and suggestions for future work, which are presented in Section 6. Appendix contains a visualization, which brings to the surface the theoretical foundation of coalition formation. Finally, interested readers would benefit from exploring the Excel spreadsheet, which helps visualize a "realistic" intersection of interests of 20 single women and 20 single men. The addendum provides a sketched outline for the evidence of some propositions.

2. Preliminaries

Five single women and five single men are ready to participate in the Singles Party. It is assumed that all participants exhibit risk-averse behavior towards dating. To cover dating bureau expenses, such as refreshments, rewards, etc., the entrance fee is set at 50€ [1]. Thus, the cashier will be at the disposal of an amount of 500€. All the guests have been kindly asked to take part in a survey, helping determine the attributes they look for in their prospective partner. Those who choose to provide this information have been promised to collect a *Box of Delights* [2] and are hereafter referred to as *participants*, while others are labeled as *dummies*, by default, and cannot participate in the game. In addition to the *delights*, promised to those willing to reveal their priorities, we continue setting the rules of payoffs in the form of incentives and mismatch compensations. However, if all participants decide to date, as no reasonable justification exists for incentives and compensations, the game terminates immediately.

We use index i for the women, and an index j for the men taking part in the dating party. Assuming that all the guests have agreed to participate in the game, there are $\{1,\ldots,i,\ldots5\}$ women and $\{1,\ldots,j,\ldots5\}$ men, resulting in $2\times5\times5$ combinations. Indeed, when priorities have been revealed, they can form two 5×5 tables, $W=\|w_{i,j}\|$, and $M=\|m_{i,j}\|$, indicating that each woman i, $i=\overline{1,5}$ revealed her priorities positioned in the rows of table W towards men as horizontal permutations w_i of numbers $\langle1,2,3,4,5\rangle$. Similarly, each man j, $j=\overline{1,5}$, revealed his priorities positioned in columns of the table M towards women as vertical permutations m_j. As can be seen in Table-1, priorities $w_{i,j}$ (numbers $\langle\overline{1,5}=1,2,3,4,5\rangle$) might repeat in both the columns of the table W and in the rows of the table M. To be sure, more than one man may prefer the same woman at priority level $w_{i,j}$, and many women, accordingly, may prefer the same man at the level $m_{j,i}$. Thus, duplets or mutual risks $r_{i,j}=w_{i,j}+m_{i,j}$ occupy the cells in table $R=\|r_{i,j}\|$.

[1] Note that red colour points at negative number.

[2] In case the Box is undesirable it will be possible to get 10€ in return.

Table1 $W + M = R$

W	M1	M_2	M_3	M_4	M_5
W_1	1	5	3	2	4
W_2	5	4	1	2	3
W_3	3	5	4	2	1
W_4	2	5	3	1	4
W_5	4	3	1	2	5

Women Priorities

M	M_1	M_2	M_3	M_4	M_5
W_1	3	4	2	1	2
W_2	1	3	4	2	4
W_3	5	2	3	4	3
W_4	4	5	1	3	1
W_5	2	1	5	5	5

Men Priorities

R	M_1	M_2	M_3	M_4	M_5
W_1	4	9	5	3	6
W_2	6	7	5	4	7
W_3	8	7	7	6	4
W_4	6	10	4	4	5
W_5	6	4	6	7	10

Mutual Duplets/Risks

Noting the assumption that all participants are risk-averse, some lucky couples with lower level of mutual risks start dating. These lucky couples will receive an incentive, such as a prepaid ticket to an event, free restaurant meal, etc. On the other hand, unlucky participants—i.e., those that did not find a partner—may claim a compensation, as only high-level mutual risk partners remained, given that the eligible participants at the low level of mutual risk have been matched.

If no one has found a suitable partner, the question is—should the party continue? Apparently, given that the original data that failed to produce matches might have not been completely truthful, it would be unwise to offer compensation in proportion to mutual risks $r_{i,j}$. Nonetheless, let us assume that the compensation equals $\frac{1}{2} r_{i,j} \cdot 10$€. In that case, couple's $[5,5]$ profit may reach 50€! Instead, the dating bureau decides to organize the game, encouraging the players to follow Rawls second principle of justice. In Table-1, the minimum—the lowest **mutual risk** among all participants—is $r_{1,4} = 3$. Following the principle, the compensation to all unlucky participants will be equal to $\frac{1}{2} r_{1,4} \cdot 10 = 15$€. This setting is also fiscally reasonable from the cashier's point of view. The balance of payoffs for all participants, will be 25€, as 50€ paid as entrance fee will be reduced by 15€ compensation amount, and additionally by 10€, i.e., inclusive of the cost of collected delights. Further on, we assume that each member of a dating couple will receive an incentive that is offered to all dating couples and is equal to double the compensation amount.

What happens when the couple $[1,4]$ decides to date? The entire table R should be dynamically transformed to reflect the fact that the participants $[1,4]$ are matched. Indeed, as the women $\{2,3,4,5\}$ and men $\{1,2,3,5\}$ can no longer count on $[1,4]$ as their potential partners, the priorities will decline, whereby the scale $\langle 1,2,3,4,5 \rangle$ dynamically shrinks to $\langle 1,2,3,4 \rangle$[3]. To reflect this, Table-1 transforms into Table-2:

[3] To highlight theoretical results of mutual risks, incitements or compensations, or whatever the scales we use, the dynamic quality of monotonic scales is the only feature fostering the birth of MS—the "monotone system." Otherwise, the MS terminology, if used in any type of serialization methods applied for data analysis, will remain sterile.

Table2 $W + M = R$

Women Priorities (W)

	M_1	M_2	M_3	M_4	M_5
W_1					
W_2	4	3	1		2
W_3	2	4	3		1
W_4	1	4	2		3
W_5	3	2	1		4

Men Priorities (M)

	M_1	M_2	M_3	M_4	M_5
W_1					
W_2	1	3	3		3
	4	2	2		2
W_4	3	4	1		1
W_5	2	1	4		4

Mutual Duplets/Risks (R)

	M_1	M_2	M_3	M_4	M_5
W_1					
W_2	5	6	4		5
W_3	6	6	5		3
W_4	4	8	3		4
W_5	5	3	5		8

The ***minimum*** mismatch compensation did not change and is still equal to 15€. However, couple's $[1,4]$ potential balance 50€+10€+2·15€ = 10€ of payoffs improves (W_1 and M_4 each receive 30€ as an incentive to date, based on the rule that it is equal to twice the mismatch compensation). For those not yet matched, the balance remains negative (in deficit) and equals 15€. On the other hand, if, for example, the couple $[3,5]$ decides to date, the balance of payoffs improves as well.

The party is over and the decisions have been made about who will date and who will leave the party without a partner. The results are passed in writing to the dating bureau. What would be the best collective decision of the participants based on the principle of "***maxima of the lowest***" in accord with the rules of singles-game?

3. Conventional stability[4]

In this section, the aim is to present the well-established solution to the singles-game by utilizing the conventional concept, called the core. First, without any warranty, it is helpful to focus on the core stability.

In order to meet this aim, the original dating party arrangement is expanded to a more general case. The game now has $n \times m$ participants, of whom n are single women $\langle 1,\ldots,i,\ldots n\rangle$ and m are single men $\langle 1,\ldots,j,\ldots m\rangle$. Some of the guests expressed their willingness to participate in the game and have revealed their priorities. Those who refused, in line with the above, are referred to as *dummy players*. All those who agreed to play the game will be arranged by default into the grand coalition $\mathscr{P}$, $|\mathscr{P}| \le n+m$. Thus, indices i, j and labels $\alpha,\ldots,\sigma \in \mathscr{P}$ are used to annotate the guests participating in the game. Only the guests in $\mathscr{P}$ are regarded as *participants*, whereas couples $[i, j]$ are referred to as $\alpha,\ldots,\sigma$. This differentiation not only helps make notations short, when needed, but can also be used in reference to an eventual match or a couple without any emphasis on gender.

In the singles-game, we focus on the participants $D \subseteq \mathscr{P}$ who are matched. Having formed a coalition, we suppose that coalition D has the power and is in a position to enforce its priorities. It is assumed that participants in D can persuade all those in $X = \mathscr{P} \setminus D$, i.e., participants that are not yet matched, to leave the party without a

[4] Terminology, which we shall use below, is somewhat conventional but mixed with our own.

partner and thus receive compensation. However, it is realistic to assume that the suppression of interests of participants' in X is not always possible. It is conceivable that, those in the coalition $D' \subseteq X$, whose interests would be affected (suppressed), will still be capable to receive as much as the participants in D. However, we exclude this opportunity, as it is better that no one expects that coalition D' can be realized concurrently with D and act as its direct competition.

Insisting on this restriction, however, we still assume that others—those participants suppressed in X—have not yet found their suitable partners and have agreed to form their own coalition, even though they could receive compensation equal to 50% of the incentives in D. A realistic situation may occur when all participants in $\mathscr{P}$ are matched, $D = \mathscr{P}$, or, in contrast, no one decides to date, $D = \varnothing$. It is also reasonable that, after revealing their priorities, some individuals might decide not to proceed with the game and will, thus, be labeled as a *dummy* player $\delta \notin \mathscr{P}$.

Among all coalitions D, we usually distinguish rational coalitions. Couple α, joining the coalition D, extracts from the interaction in the coalition a benefit that satisfies $\alpha \in D$. In the singles-game, we anticipate that the extraction of benefits, i.e., the incentives and mismatch compensations, strictly depend on the membership—couples in D or participants of coalition X. Using the coalition membership $D \subseteq \mathscr{P}$, we can always construct a payoff x to all participants $\mathscr{P}$, i.e., we can quantify the positions of all participants. The inverse is also true. Given a payoff x, it is easy to establish which couple belongs to the coalition D and identify those belonging to the coalition $X = \mathscr{P} \setminus D$. We label this fact as D_x. Recall that couples of the coalition D_x receive an incentive to date, which is equal to the double amount of the mismatch compensation. Thus, the allocation D_x may provide an opportunity for some participants $\sigma \in \mathscr{P}$ to start, or initiate, new matches, thus moving to better positions. We will soon see that, while the best positions induced by special coalitions $\mathscr{N}$, called the nuclei, have been reached, this movement will be impossible to realize. [5]

The inability of players to move to better positions by "pair comparisons" is an example of stability. In the work "Cores of Convex games", convex games have been studied (Shapley, 1971); these are so-called games with a non-empty core, where similar type of stability exists. The core forms a convex set of end-points (*imputations*) of a multidimensional octahedron, i.e., a collection of available payoffs to all players. Below, despite the players' asymmetry with respect to $D_x = \mathscr{P} \setminus X$, we focus on their payoffs driving their collective behavior as participants $\mathscr{P}$ to form a coalition D_x, $D_x \subseteq \mathscr{P}$; here, $\overline{X} = D_x$ is called an anti-coalition to X.

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[5] Our terminology is unconventional in this connection.

In contrast to individual payoffs improving or worsening the positions of participants, when playing a coalition game, the total payment to a coalition X as a whole is referred to the characteristic function $v(X) > 0$. In classical cooperative game theory, the payment $v(X)$ to coalition X is known with certainty, whereby the variance $v(X) - v(X \setminus \{\sigma\})$ provides a marginal utility $\pi(\sigma, X)$. Inequality $\pi(\alpha, X \setminus \{\sigma\}) \leq \pi(\alpha, X)$ of the scale of risks expresses a monotonic decrease (increase) in marginal utilities of the membership for $\alpha \in X$. The monotonicity is equivalent to the supermodularity $v(X_1) + v(X_2) \leq v(X_1 \cup X_2) + v(X_1 \cap X_2)$ (Nemhauser et al., 1978). Thus, any characteristic function $v(X)$, payment on which is built according to the scale of risks, is supermodular. The inverse submodularity was used to find solutions of many combinatorial problems (Edmonds, 1970; Petrov and Cherenin, 1948). In general, such a warranty cannot be given.

Recall that we eliminated all rows and columns in tables $W = \|w_{i,j}\|$, $M = \|m_{i,j}\|$ in line with $\overline{X} = D_x$. Table $R = \|\pi(\alpha, X) = w_{i,j}(X) + m_{i,j}(X)\|$, $\alpha = [i, j] \in X$ reflects the outcome of shrinking priorities $w_{i,j}$, $m_{i,j}$ when some $\sigma \in \overline{X}$ have found a match and have formed a couple. Priorities $w_{i,j}$, $m_{i,j}$ are consequently decreasing. Given in the form of characteristic function, e.g., the value $v(X) = \sum_{\alpha \in X} \pi(\alpha, X)$ sets up a coalition game.[6] An imputation for the game $v(X)$ is defined by a $|\mathcal{P}|$-vector fulfilling two conditions: **(i)** $\sum_{\alpha \in \mathcal{P}} x_\alpha = v(\mathcal{P})$, **(ii)** $x_\alpha \geq v(\{\alpha\})$, for all $\alpha \in \mathcal{P}$. Condition (ii) clearly stems from repetitive use of monotonic inequality $\pi(\alpha, X \setminus \{\sigma\}) \leq \pi(\alpha, X)$.

A significant shortcoming of the cooperative theory given in the form of the characteristic function stems from its inability to specify a particular imputation as a solution. However, in our case, such imputation can be defined in an intuitive way. In fact, the concept of risk scale determines a popularity index of players. More specifically, the lower the risk of engagement $\pi(\alpha, X)$ of $\alpha \in X$, the more reliable the couple's α coexistence is. Therefore, we set up a popularity index p_i of a woman i among men in the coalition X as number $p_i(X) = \sum_{j \in X} m_{i,j}$. The index number p_j of a man j among women, accordingly, is given by $p_j(X) = \sum_{i \in X} w_{i,j}$. We intend to redistribute the total payment $v(X)$ in proportion to the components of the vector $p(X) = \langle p_i(X), p_j(X) \rangle$, or as the vector $p(X)$. Hereby we can prove, owing to monotonicity of the scale of priorities, that the payoffs in imputation $p(\mathcal{P})$ cannot be improved by any coalition $X \subset \mathcal{P}$. Therefore, the game solution, among popularity indices, will be the only imputation $p(\mathcal{P})$. In other words, popularity indices core of the cooperative game $v(X)$ consists of only one point $p(\mathcal{P})$.

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[6] $v(X) = |X|^2 \cdot (|X| + 1)$. Check that $v(\mathcal{P}) = 150$ for 5×5-game, or use the Table-1.

In line with the terminology used above, we draw the readers' attention to the fact that the singles-game considered next is not a game given in the form of a characteristic function. The above discussion was presented as the foundation for the course of further investigation only.

4. Concept of a kernel

In the view of "monotone system" (Mullat, 1971-1995) exactly as in Shapley's convex games, the basic requirement of our model validity emerges from an inequality of monotonicity $\pi(\alpha,X\setminus\{\sigma\})\le\pi(\alpha,X)$. This means that, by eliminating an element σ from X, the utilities (weights) on the rest will decline or remain the same. In particular, a class of monotone systems is called **p**-monotone (Kuznetsov et al., 1982, 1985), where the ordering $\langle\pi(\alpha,X)\rangle$ on each subset X of utilities (weights) follows the initial ordering $\langle\pi(\alpha,\mathcal{W})\rangle$ on the set $\mathcal{W}$. The decline of the utilities on **p**-monotone system does not change the ordering of utilities on any subset X. Thus, serialization (greedy) methods on **p**-monotone system might be effective. Behind a **p**-monotone system is the fact that an application of Greedy framework can simultaneously accommodate the structure of all subsets $X\subset\mathcal{W}$. Perhaps, for different reasons, many will argue that **p**-monotone systems are rather simplistic and fail to compare to the serialization method. Nonetheless, many economists, including Narens and Luce (1983), almost certainly, will point out that subsets X of **p**-monotone systems *perform* on interpersonally compatible scales.

An inequality $F(X_1\cup X_2)\ge min\langle F(X_1),F(X_2)\rangle$ holds for real valued set function $F(X)=min_{\alpha\in X}\pi(\alpha,X)$, referred to as quasi-convexity (Malishevski, 1998). We observed monotone systems, which we think is important to distinguish. The system is non quasi-convex when two coalitions contradict the last inequality. We consider such systems as non-quasi-convex, which applies to the singles-game case.

The ordering of priorities in singles-games—i.e., what men look for in women, and vice versa—remain intact within an arbitrary coalition X. However, in these systems, the ordering of mutual risks $\|r_{i,j}\|$ on grand coalition $\mathcal{P}$ does not necessarily hold for some $X\subset\mathcal{P}$. Contrary to initial ordering on $R(\mathcal{P})=\|\pi(\alpha,\mathcal{P})=r_{i,j}\|$, the ordering of mutual risks on $R(X)=\|\pi(\alpha,X)\|$ may be inverse of the ordering on $R(\mathcal{P})$ for some couples. In that case, e.g., the ordering of two couples' mutual risks can turn "upside down" while the risk scale is shrinking compared to the original ordering on the grand coalition $\mathcal{P}$. Thus, in general, the mutual risks scale is not necessarily interpersonally compatible. In other words, interpersonal incompatibility of this risk scale radically differs from the **p**-monotone systems. This difference became apparent when it was no longer possible to find a solution using Greedy type framework of so-called defining chain algorithm—i.e., the monotone system was non-quasi-convex. Before proceeding with the formal side of these processes, it is informative to understand the nature of the problem.

Definition 1 *By kernel coalition we call a coalition* $\mathcal{K} \in arg\,max_{X \subseteq \mathcal{P}} F(X)$; $\{\mathcal{K}\}$ *is the set of all kernels.*

Recalling the main quality of defining a chain—a sequence of elements of a monotone system—it is possible to arrange the elements $\alpha \in \mathcal{W}$, i.e., the couples $\alpha \in \mathcal{P}$ of players by a sequence $\langle \alpha_1,...,\alpha_k \rangle$, $k = \overline{1,n}$. The sequence follows the lowest risk ordering in each step k corresponding to sequence of coalitions $\langle H_k \rangle$, $H_1 = \mathcal{P}$, $H_{k+1} \leftarrow H_k \setminus \{\alpha_k\}$, $\alpha_k = arg\,min_{\alpha \in H_k} \pi(\alpha, H_k)$. Given any arbitrarily coalition $X \subseteq \mathcal{P}$, we say that the defining sequence obeys the left concurrence quality if there exists a superset H_t such that $H_t \supseteq X$, $t = \overline{1,k}$, where the first element $\alpha_t \in H_t$ to the left in the sequence $\langle \alpha_1,...,\alpha_k \rangle$ belongs to the set X, $\alpha_t \in X$ as well. On the condition that the element α_t is not a member of the superset $\mathcal{H} = \bigcup\{\mathcal{K} \in arg\,max_{X \subseteq \mathcal{P}} F(X)\}$ including all kernels $\mathcal{K}$, $\alpha_k \notin \mathcal{H}$, we observe that $\pi(\alpha_t, X) < \pi(\alpha_t, H_t)$. Hereby, we can conclude that $F(X) \leq \pi(\alpha_t, H_t)$ is strictly less than the global maximum of the set function $F(X) = min_{\alpha \in X} \pi(\alpha, X)$. The left concurrence quality guarantees that the sequence can potentially be used for finding the largest kernel $\mathcal{H}$. Due to non-quasi-concavity, the left concurrence quality is no longer valid. Eliminating a couple $\alpha_k = [i, j]$, see above, we delete the row i and the column j in the mutual risks table R. Thus, the operation $H_{k+1} \leftarrow H_k \setminus \{\alpha_k\}$ is not an exclusion of a couple $\alpha_k \in H_k$, given that the couple $\alpha_k = [i, j]$ is about to start dating, but rather an exclusion of adjacent couples α in $[i,*]$-row and $[*, j]$-column. We annotate the engagement as $H_{k+1} \leftarrow H_k - \alpha_k$ or as an equal notation $D_{k+1} \leftarrow D_k + \alpha_k$.

In conclusion, note, once again, that, despite the properties of monotone system remaining intact, the chain algorithm, assembling the defining sequence of elements $\alpha \in \mathcal{P}$, cannot guarantee the extraction of the supposedly largest kernel $\mathcal{H}$, particularly in the form given by Kempner et al. (2008). Thus, we need to employ special tools for finding the solution. To move further in this direction, we are ready to formulate some propositions for finding kernels $\mathcal{K}$ by branch and bound algorithm types.

The next step will require a modified variant of imputation (Owen, 1982). We define an imputation as the outcome connected to the singles-game in the form of a $|\mathcal{P}|$-vector of payoffs to all participants. More specifically, the outcome is a $|\mathcal{P}|$-vector, where each partner in a couple $\sigma \in X$ receives the ***lowest*** mismatch compensation $F(X)$, whereas each partner in the couple $\sigma \notin X$ belonging to the anti-coalition $\overline{X} = D_x$ receives the incentive to date, which is equal to twice that amount, i.e., $2 \cdot F(X)$, cf. Tables 3,4. The concept of outcome (imputation) in this form is not common because the amount to be claimed by all participants is not fixed and equals $|\mathcal{P}| + F(X) \cdot \left(|X| + 2 \cdot |\overline{X}|\right)$. Thus, it is likely that participants will fail to reach an understanding, and will claim payoffs obtaining less than available total amount $(n+m) \cdot 50$€. The situation, in contrast, when participants will claim more than total amount, is also conceivable.

Any coalition X induces a $|\mathcal{P}|$-vector $x = \langle x_\sigma \rangle$ as an outcome x:[7]

$$x_\sigma = \begin{cases} 2 + F(X) \text{ if } \sigma \in X, \\ 2 \cdot (1 + F(X)) \text{ if } \sigma \notin X. \end{cases} \rightarrow \sum_{\sigma \in \mathcal{P}} x_\sigma = |\mathcal{P}| + F(X) \cdot \left(|X| + 2 \cdot |\overline{X}|\right).$$

In this case, x_σ is a quasi-imputation.

This definition of *outcome* is used later, adapting the concept of the quasi-imputation for the purpose of the singles-game. We say that an arbitrary coalition X induces an outcome x. Computed and prescribed by coalition X, the components of x consist of two distinct values $F(X)$ and $2 \cdot F(X)$. Participants $\sigma \in X$ could not form a couple, while participants $\sigma \in D_x$ were able to match. Recall that the notation for $\overline{X}$ is also used as a mixed notation for dating couples D_x.

Before we move further, we will try to justify our mixed notation $\overline{X}$. Although a coalition $\overline{X} = D_x$ uniquely defines both those D_x among participants $\mathcal{P}$ who went on dating, and those $X = \mathcal{P} \setminus D_x$ who did not, the coalition $\overline{X}$ does not specifically indicate matched couples. In contrast, using the notation D_x, we indicate that all participants in D_x are matched, whereas a couple $\sigma \in D_x$ also indicates an individual decision how to match. More specifically, this annotation represents all men and all women in D_x standing in line facing one member of the opposite sex, with whom they are matched. However, any matching or engagement among couples belonging to D_x, or whatever matches are formed in D_x, does not change the payoffs x_σ valid for the outcome x. In other words, each particular matching D_x induces the same

[7] Further, we follow the rule that capital letters represent coalitions $X, Y, \ldots, \mathcal{K}, \mathcal{H}, \ldots$ while lower-case letters $x, y, \ldots, \boldsymbol{k}, \boldsymbol{h}, \ldots$ represent outcomes induced by these coalitions.

outcome x. Decisions in D_x with respect to how to match provide an example of individual rationality, while the coalition D_x formation, as a whole, is an example of collective rationality. Therefore, in accordance with payoffs x, the notation D_x subsumes two different types of rationality—the individual and the collective rationality. In that case, the outcome x accompanying D_x represents the result of a partial matching of participants $\mathcal{P}$. Propositions below somehow bind the individual rationality with the collective rationality.

One of the central issues in the coalition game theory is the question of the possible formation of coalitions or their accessibility, i.e., the question of coalition feasibility. While it is traditionally assumed that any coalition $X \subseteq \mathcal{P}$ is accessible or available for formation, such an approach is generally unsatisfactory. We will try to associate this issue with a similar concept in the theory of monotone systems. The issue of accessibility of subsets $X \subset \mathcal{W}$ in the literature of monotone systems has been considered not only in the context of the totality $2^{\mathcal{W}}$ of its subsets $X \in 2^{\mathcal{W}}$ but also with respect to special collections of subsets $\mathcal{F} \subset 2^{\mathcal{W}}$. A singleton chain α_t adding elements step-by-step, starting with the empty set $\varnothing$, can, in principle, access any set $X \in \mathcal{F}$, or access the set X by removing the elements starting with the grand set $\mathcal{W}$—so called upwards or downwards accessibility.

Definition 2 *Given coalition $X \subseteq \mathcal{P}$, where $\mathcal{P}$ is the grand coalition, we call the collection of pairs $C(X) = \{arg\,min_{\alpha \in X}\, \pi(\alpha, X)\}$ naming $C(X)$ as best potential couples, capable of matching with the lowest mutual risk, within the coalition X.*

Consider a coalition D_x, generated by the formation by a chain of steps $D_{k+1} \leftarrow D_k + \langle \alpha_k \rangle$. Let $X_1 = \mathcal{P}$, $X_k = \mathcal{P} \setminus D_k$, where D_k are participants trying to match during the step k; $C(X_k)$ are couples in X_k with the lowest mutual risk among couples not yet matched in steps $k = \overline{1,n}$, $X_{n+1} = \varnothing$. Coalitions in the chain $X_{k+1} = X_k - \alpha_k$ are arranged after the rows and columns, indicated by couple α_k, have been removed from W, M and R. Mutual risks R have been recalculated accordingly.

Definition 3 *Given the sequence $\langle \alpha_1, \dots, \alpha_k \rangle$ of matched couples, where $X_1 = \mathcal{P}$, $X_{k+1} = X_k - \alpha_k$, we say that coalition $D_x = \overline{X} = \mathcal{P} \setminus X$ of matched (as well as X of not yet matched) participants is feasible, when the chain $\langle X_1, \dots, X_{k+1} = X \rangle$ complies with the rational succession $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$. We call the outcome x, induced by sequence $\langle \alpha_1, \dots, \alpha_k \rangle$, a feasible payoff, or a feasible outcome.*

Proposition 1 *The rational succession rationality necessarily emerges from the condition that, under the coalition D_x formation a couple α_k does not decrease the payoffs of couples $\langle \alpha_1,...\alpha_{k-1} \rangle$ formed in previous steps.*

The accessibility or feasibility of coalition D_x formation offers convincing interpretation. In fact, the feasibility of coalition D_x means that the coalition can be formed by bringing into it a positive increment of utilities to all participants $\mathscr{P}$, or by improving the position of existing participants having already formed a coalition when new couples enter the coalition in subsequent steps. We claim that, in such a situation, coalitions are formed by rational choice. The rational choice $C(X)$ satisfies so-called heritage or succession rationality described by Chernoff (1954), Sen (1970), and Arrow (1959). Below, we outline the heritage rationality in the form suitable for visualization.

The proposition states that, in matches, the individual decisions are also rational in a collective sense only when all participants in D_x individually find a suitable partner. We can use different techniques to meet the individual and collective rationality by matching all participants only in D_x, which is akin to the stable marriage procedure (Gale & Shapley, 1962). In contrast, the algorithm below provides an optimal outcome/payoff accompanied by partial matching only—i.e., only matching some of participants in $\mathscr{P}$ as participants of D_x; once again, this is in line with the Greedy type matching technique.

Proposition 2 *The set $\{\mathcal{K}\}$ of kernels in the singles-game arranges feasible* coalitions. *Any outcome κ induced by a kernel $\mathcal{K} \in \{\mathcal{K}\}$ is feasible.*

At last, we are ready to focus on our main concept.

Definition 4 *Given a pair of outcomes x and y, induced by coalitions X and Y, an outcome y dominates the outcome x, $x \prec y$:*

(i) $\exists S \subseteq \mathscr{P} | \forall \sigma \in S \to x_\sigma < y_\sigma$, (ii) the outcome y is feasible.

Condition (i) states that participants/couples $\sigma \in S \subset \mathscr{P}$ receiving payoffs x_σ can break the initial matching in D_x and establish new matches while uniting into D_y. Alternatively, some members of X, i.e., not yet matched participants in S, can find suitable partners amid participants in D_y, or, even their compensations in Y may be higher than their incentives in x. Thus, by receiving y_σ instead of x_σ the participants belonging to S are guaranteed to improve their positions. The interpretation of the condition (ii) is obvious. Thus, the relation $x \prec y$ indicates that participants in S can cause a split (bifurcation) of D_x, or are likely to undermine the outcome x.

Definition 5 *A kernel* $\mathcal{N} \in \{\mathcal{K}\}$ *minimal by inclusion is called a nucleus—it does not include any other proper kernel* $\mathcal{K} \subset \mathcal{N}$ *:* $\mathcal{K} \not\subset \mathcal{N}$ *is true for all* $\mathcal{K} \neq \mathcal{N}$ *.*

Proposition 3 *The set* $\{\boldsymbol{n}\}$ *of outcomes, induced by nuclei* $\{\mathcal{N}\}$*, arranges a quasi-core of the singles-game. Outcomes in* $\{\boldsymbol{n}\}$ *are non-dominant upon each other, i.e.,* $\boldsymbol{n} \prec \boldsymbol{n}'$*, or* $\boldsymbol{n} \succ \boldsymbol{n}'$ *is false.* Thus, the quasi-core is internally stable.

The proposition above clearly indicates that the concept of internal stability is based on "pair comparisons" (binary relation) of outcomes. The traditional solution of coalition games recognizes a more challenging stability, known as *NM* solution, which, in addition to the internal stability, demands external stability. External stability ensures that any outcome x of the game outside *NM*-solution cannot be realized because there is an outcome $\boldsymbol{n} \in \{\boldsymbol{n}\}$, which is not worse for all, but it is necessarily better for some participants in x. Therefore, most likely, only the outcomes $\boldsymbol{n}$ that belong to *NM*-solution might be realized. The disadvantage of this scenario stems from the inability to specify how it can occur. In contrast, in the singles-game, we can define how the transformation of one coalition to another takes place, namely, only along feasible sequence of couples. However, it may happen that for some coalitions X outside the quasi-core $\{\mathcal{N}\}$, feasible sequence may stall unable to reach any nucleus $\mathcal{N} \in \{\mathcal{N}\}$, whereby starting at X the quasi-core is feasibly unreachable. This is a significant difference with respect to the traditional *NM*-solution.

5. Finding the quasi-core

In general, when using Greedy type algorithms, we gradually improve the solution by a local transformation. In our case, a contradiction exists because nowhere is stated that local improvements can effectively detect the best solution—the best outcome or payoffs to all players. The set of best payoffs, as we already established above, arranges a quasi-core of the game. Usually, finding the core in the conventional sense is a NP-hard task, as the number of "operations" increases exponentially, depending on the number of participants. In the singles-game, or in almost all other types of coalition games, we observe an extensive family of subsets constituting traditional core imputations. Even if it is possible to find all the payoff vectors in the core, it is impractical to do so. We thus posit that it is sufficient to find some feasible coalitions belonging to the quasi-core and the payoffs induced by these coalitions.

This can be accomplished by applying a procedure of *strong improvements* of payoffs, and several *gliding procedures*, which do not worsen the players' positions under coalition formation. Indeed, based on rationality, known as the rational succession, Definition 3, it is not rational in some situations to use the procedure of strong improvements, as these do not exist. However, using gliding procedures, we can move forward in one of the promising directions to find payoffs not worsening the outcome. Experiments conducted using our polynomial algorithm show that, while using a mixture of improvement procedure and gliding procedures, combined with the succession condition, one can take the advantage of backtracking strategy, and might find feasible payoffs of the singles-game belonging to the quasi-core.

We use five procedures in total—one improvement procedure and four variants of gliding procedures. Combining these procedures, the algorithm below is given in a more general form. While we do not aim to explain in detail how to implement these five procedures, in relation to rational succession, it will be useful to explain beforehand some specifics of the procedures because a visual interaction is best way to implement the algorithm.

In the algorithm, we can distinguish two different situations that will determine in which direction to proceed. The first direction promises an improvement in case the couple $\alpha \in X$ decides to match. We call the situation when $C(X-\alpha) \cap C(X) = \varnothing$ as a potential improvement situation. Otherwise, when $C(X-\alpha) \cap C(X) \neq \varnothing$, it is a potential gliding direction. Let $CH(X)$ be the set of rows $C(X)$, the horizontal routes in the table R, which contain the set $C(X)$. By analogy $CV(X)$ represents the vertical routes, the set of columns, $C(X) \subseteq CH(X) \times CV(X)$. To apply our strategy upon X, we distinguish four cases of four non-overlapping blocks in the mutual risk table $R: CH(X) \times CV(X)$; $CH(X) \times \overline{CV(X)}$; $\overline{CH(X)} \times CV(X)$; $\overline{CH(X)} \times \overline{CV(X)}$.

Proposition 4 *An improvement in payoffs for all participants in the singles-game may occur only when a couple* $\alpha \in X$ *complies with the potential improvement situation in relation to the coalition* X, *the case of* $C(X-\alpha) \cap C(X) = \varnothing$. *The couple* $\alpha \in X$ *is otherwise in a potential gliding situation.*

The following algorithm represents a heuristic approach to finding a nucleus $\boldsymbol{n}$ among nuclei $\{\mathcal{N}\}$ of the singles-game.

Input Build the mutual risks table, $R = W + M$ —a simple operation in Excel spreadsheet. Recall the notation $\mathcal{P}$ of players as the game participants. Set $k \leftarrow 1$, $X \leftarrow \mathcal{P}$ in the role of not yet matched participants, i.e., as players available for potential matching**.** In contrast to the set X, allocate indicating by $D_x \leftarrow \varnothing$ the initial status of matched participants.

Do Step up: **S** Find a match $\alpha_k \in CH(X) \times CV(X)$, $D_x \leftarrow D_x + \alpha_k$, such that $F(X) < F(X - \alpha_k)$, $X \leftarrow X - \alpha_k$, $X_k = X$, $k = k+1$, otherwise *Track Back*.

Gliding: **D** Find a match $\alpha_k \in CH(X) \times CV(X)$, $D_x \leftarrow D_x + \alpha_k$, such that $F(X) = F(X - \alpha_k)$, $X \leftarrow X - \alpha_k$, $X_k = X$, $k = k+1$, otherwise *Track Back*.

F Find a match $\alpha_k \in CH(X) \times \overline{CV(X)}$, $D_x \leftarrow D_x + \alpha_k$, such that $F(X) = F(X - \alpha_k)$, $X \leftarrow X - \alpha_k$, $X_k = X$, $k = k+1$, otherwise *Track Back*.

G Find a match $\alpha_k \in \overline{CH(X)} \times CV(X)$, $D_x \leftarrow D_x + \alpha_k$, such that $F(X) = F(X - \alpha_k)$, $X \leftarrow X - \alpha_k$, $X_k = X$, $k = k+1$, otherwise *Track Back*.

H Find a match $\alpha_k \in \overline{CH(X)} \times \overline{CV(X)}$, $D_x \leftarrow D_x + \alpha_k$, such that $F(X) = F(X - \alpha_k)$, $X \leftarrow X - \alpha_k$, $X_k = X$, $k = k+1$, otherwise *Track Back*.

Loop Until no couples to match can be found in accordance with cases **S, D, F, G** and **H.**

Output The set D_x has the form $D_x = \langle \alpha_1, ..., \alpha_k \rangle$. The set $\mathcal{N} = \mathcal{P} \setminus D_x$ represents a nucleus of the game while the payoff $\boldsymbol{n}$ induced by $\mathcal{N}$ belongs to the quasi-core.

In closing, it is worth noting that a technically minded reader would likely observe that coalitions X_k are of two types. The first case is $X \leftarrow X - \alpha_k$ operation when the mismatch compensation increases, i.e., $F(X_k) < F(X_k - \alpha_k)$. The second case occurs when gliding along the compensation $F(X_k) = F(X_k - \alpha_k)$. In general, independently of the first or the second type, there are five different directions in which a move ahead can proceed. In fact, this poses a question—in which sequence couples α_t should be selected in order to facilitate the generation of the *sequence* $D_x = \langle \alpha_1,...,\alpha_k \rangle$? We solved the problem for singles-games underpinning our solution by backtracking. It is often clear in which direction to move ahead by selecting improvements, i.e., either a strict improvement by **s)** or gliding procedures though **d)**, **f)**, **g)** or **h)**. However, a full explanation of backtracking is out of the scope of our current investigation. Thus, for more details, one may refer to similar techniques, which effectively solve the problem (Dumbadze, 1989).

6. Conclusions

The uniqueness of singles-game lies in the dynamic nature of priorities. As the construction of the matching sequence proceeds, priorities dynamically shrink, and finally converge at one point. Dynamic transformation, or the monotonic (dynamic) nature of priorities, enabled constructing a game based on so-called monotone system, or MS. One disadvantage behind the use of the MS-system is its drawback in the respective interpretation of the analysis results. More specifically, when the process of extracting the core terminates, the interpretation requires further corrections. However, with regards to the choice of the best variants, i.e., the choice of the best matches in the singles-game, the paper reports a scalar optimization in line with "***maxima of the lowest***" principle, or rather an optimal choice of partial matching. This view opens the way to consider the best partial matching as the choice of the best variants—alternatives—and to explore the matching process from the perspective of a choice problem.

Usually, when trying to analyze the results, a researcher must rely on the common sense. Therefore, applying the well-known and well thought out concepts and categories that have been successfully applied in the past, we can move forward in the right direction. Our advantage was that this relation was found, and was transformed into a shape similar to the core, which is known concept in the theory of stability of collective behavior, e.g., in the theory of coalitional games.

Irrespective of the complexity of intersections in the interests of players, deftly twisted rules for compensations in unfortunate circumstances, incitements, etc., singles-game, as it seems, makes a point. However, this is not enough in social sciences, especially in economics, when a formal scheme rarely depicts the reality, e.g., the difference in political views and positions of certain groups of interest, etc. Perhaps, the individual components of the game will still be helpful in moving closer to answering the question of what is right or wrong, or what is good and what is bad, which would be a fruitful path to explore in future studies of this type.

Appendix

Visualization

Recall that, in the singles-game, the input to the algorithm presented in the main paper contains two tables: $W = w_{i,j}$ —priorities w_i the women specify with the respect to the characteristics the men should possess, in the form of permutations of numbers $\overline{1,n}$ in rows, and the table $M = m_{j,i}$ —priorities m_j the men specify with the respect to the characteristics the women should possess, in the form of permutations of numbers $\overline{1,m}$ in columns. These tables, and tabular information in general, are well-suited for use in Excel spreadsheets that feature calculation, graphing tools, pivot tables, and a macro programming language called VBA—Visual Basic for Applications.

A spreadsheet was developed in order to present our idea visually, i.e., the search for nuclei of the singles-game, and the stable coalitions with outcomes belonging to the quasi-core induced by these coalitions. The spreadsheet takes for granted the Excel functions and capabilities. Tables W, M and R of 20×20 dimensions can be downloaded from http://www.datalaundering.com/download/singles-game.xls. We first provide the user with the list of macros written in VBA. Then, we supply tables W, M and R extracted from the spreadsheet by comments. We also hope that the spreadsheet exercise will be useful in enhancing the understanding of our work. In particular, we focus on the technology of backtracking, given by macros **TrackR** and **TrackB**.

The list of macro-programming routines is in line with the steps of the algorithm presented in Section 5.

- **CaseS.** Ctrl+**s** Trying to move by improvement along the block $CH(X) \times CV(X)$ of cells [i,j] by"<" operator in order to find a new match at the strictly higher level. [8]
  find a
- **CaseD.** Ctrl+**d** Trying to move while gliding along the block $CH(X) \times CV(X)$ of cells [i,j] by "<=" operator in order to find a new match at the same or higher level.
- **CaseF.** Ctrl+**f** Trying to move while gliding along the block $CH(X) \times \overline{CV(X)}$ of cells [i,j] by "<=" operator in order to find a new match at the same or higher level.
- **CaseG.** Ctrl+**g** Trying to move while gliding along the block $\overline{CH(X)} \times CV(X)$ of cells [i,j] by "<=" operator in order to find a new match at the same or higher level.
- **CaseH.** Ctrl+**h** Trying to move while gliding along the block $\overline{CH(X)} \times \overline{CV(X)}$ of cells [i,j] by "<=" operator in order to find a new match at the same or higher level.

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[8] CH —cells in horizontal direction, CV —cells in vertical direction

V1. Spreadsheet layout

There are 20 single women and 20 single men attending the party, i.e., $n,m=20$. Three tables are thus available: The Pink table ***W***—women's priorities; The Blue table ***M***—men's priorities, and the Yellow table ***R***—the mutual risks table. The column to the right of the table ***R*** lists all women $i=\overline{1,20}$ showing $min_{j=\overline{1,20}} r_{i,j}$ level of risk of couples $[i,*]$. The row down of the bottom of table ***R*** lists all men $j=\overline{1,20}$ showing $min_{i=\overline{1,20}} r_{i,j}$ level of risk of couples $[*,j]$. In the right hand bottom corner cell, the lowest $min_{i=\overline{1,20},j=\overline{1,20}} r_{i,j} = F(X)$ level of risk over the whole table ***R*** is given. Notice that the green cells in the table ***R*** visually represent the effect of $arg\,min_{i=\overline{1,20},j=\overline{1,20}} r_{i,j}$ operation. Actually, the green cells visualize the choice operator $C(X)$. Arrays V24:AO25 and V26:AO26 will be implemented in the process of generating the matching sequence together with the levels of risk associated by this sequence. The players' balance of payoffs occupies the array V31:AO32. Some cells reflecting the *state of finances* of cashier are located below, in the array AP34:AP44. Cells in row-**1** and column-**A** contain the guests' labels. We use these labels in all macros.

V2. Functional test

The spreadsheet users are invited first to perform a functional test, in order to become familiar with the effects of **ctrl-keys** attached to different macros. Calculations in Excel can be performed in two modes, ***automatic*** and ***manual***. However, it is advisable to choose properties and set the calculus in the manual mode, as this significantly speeds up the performance of our macros.

The actions that can be taken if something goes wrong are listed below.

- **Originate.** [Ctrl+**o**] Perform the macro by Ctrl+**o**, and then use Ctrl+**b**. This macro restores the original status of the game saved by the BacKup, i.e., saved by ctrl-**k**.
- **RandM.** [Ctrl+**m**] Perform the macro by Ctrl+m. It randomly rearranges columns of **M**en's priority table **M** by random (permutations). Notice the effect upon men's priority table **M**.
- **RandW.** [Ctrl+**w**] Perform the macro by Ctrl+w. It randomly rearranges rows of **W**omen's priority table **W** by random (permutations). Notice the effect upon women's priority table **W**.
- **Proceed.** [Ctrl+**e**] While procEeding with macros Rand**M** and Rand**W,** the macro is using random permutations for men and women until it generates the priority tables **M** and **W** with minimum mutual risk equal to 4.
- **Dummy.** [Ctrl+**u**] This macro is removing from the list of participants those guests that do not wish to play the game, or who decide not to pursue the dating. We call them d**U**mmy players. Activate the row-1, or column-A by pointing at man $\mathbf{m}_{\#\#}$, or woman $\mathbf{w}_{\#\#}$ and then perform Ctrl+**u** excluding the chosen guests from playing the game.

- **MCouple.** [Ctrl+**a**] Try to m**A**tch [ctrl+**a**] a couple by pointing at the cell in the upper block: pink color to the left (or yellow to the right) in the row $\mathbf{w_i}$ (corresponding to a woman) and the column $\mathbf{m_j}$ (corresponding to a man).
- **TrackR.** [Ctrl+**r**] Visualizes Tracking forwa**R**d. Memorizes the status of *Women*-**W** and *Men*-**M** priorities to be restored by **TrackB** macro. The effect of this macro is invisible. It can be used whenever it is appropriate to save the active status of all tables and all the arrays necessary to restore the status by **TrackB** macro. Only when the search for quasi-core coalitions is performed manually, the effect of macros is visible.
- **TrackB.** [Ctrl+**b**] Visualizes Tracking **B**ack. Restores the status of *Women*-**W** and *Men*-**M** priorities memorized by **TrackR** macro.
- **Happiness** [Ctrl+**p**] The macro calculates an index of ha**P**piness using the initial risks table.
- **Coalition** [Ctrl+**n**] The macro rebuilds the matching coalitio**N** following the coalition matching list previously transferred into area "AV24:AO25".
- **Chernoff** [Ctrl+**q**] Useful when indicating **by red font** the status of the Choice Operator $C(X) = \{arg\,min\}$. Using this macro will help to confirm the validity of the Succession Operator. To clear the status, use Ctrl+**l**.

V3. Extracting nuclei of the game

We came closer to the goal of our visualization, where we visually demonstrate the main features of the theoretical model of the game by example. Generating the matching sequence, which is performed in a stepwise fashion, constitutes the framework of the theory. At each step, to the right side of the sequence generated in the preceding steps, we add a couple found by one of the macros CaseS, CaseD,…, CaseH, i.e., a couple that has decided to date. This process is repeated until all participants are matched, and the sequence is complete. One can easily verify that, the levels of risk initially increase, and decline towards the end. This single $\cap$-peakedness is a consequence of the levels of mutual risk monotonicity $\pi(\alpha, H \setminus \{\sigma\}) \leq \pi(\alpha, H)$. Indeed, recall that risk levels are recalculated after each match. With the proviso of recommendations in our heuristic algorithm, see above, due to the recalculation, the priority scales will "shrink" or "pack together", as only not yet matched participants remain. Let us try to generate a Matching Sequence using macros: CaseS, CaseD, CaseF,… . The data, e.g., will occupy the array V24:O28.

Table3	Couple nr.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Row 1	W-matched	19	10	1	6	4	17	5	2	11	18	20	8	3	9	15	12	7	13	16	14
Row 2	M-matched	5	9	10	17	15	3	6	13	14	4	1	11	2	12	8	19	16	18	7	20
Row 3	Risk levels	3	3	4	5	6	6	6	6	6	6	6	6	6	5	4	3	3	2	2	0
Row 4	W-payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
Row 5	M-payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

Observe that, starting with the couple **no. 14**, we can no longer use macros of our heuristic algorithm. Couples **no. 1-13** represent a nucleus ***n*** of the game. Thus, we can continue generating the sequence only by manual macro **MCouple**—Ctrl+a.

In Table-3, in the Matching Sequence of length 20, $k=\overline{1,20}$, we labeled couple $[i,j]$ by α using notation α_k. Together with levels of mutual risks in row 3, the rows 1,2 correspond to the sequence $\langle \alpha_k \rangle$. Compensations and incentives for dating are not payable at all, and only the costs of delight*s* (each worth 10€) occupy rows 4,5. Notice that, in accordance with single $\cap$-peakedness, the ***lowest*** levels of risk first increase starting at 3, and after reaching 6, starting at couple **no. 13**, they start declining down to 0. For couple **no. 3**, risks jump from 4 to 5, while, for couple **no. 4**, they increase from 5 to 6.

Let us look at Table-4, where only 13 matches are accomplished, i.e., all columns to right including the couple **no. 14** are empty. Table-4 visualizes the nucleus below. Pink and Blue colors mark those who decided to date, while **Yellow** marks those who have not yet taken their decisions. Hereby, **Yellow** participants occupying rows 4-5 will mark the participants of a nucleus coalition—a coalition inducing payoffs as incentives and mismatch compensations to all 40 participants—20 women and 20 men. The payoffs 40€ and 70€ corresponding to the nucleus make up the outcome. The balance of the outcome —the total amount of 2000€ as entrance fees minus payoffs 2380€ —is not in cashier's favor.

Table4	Couple nr.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*							
Row 1	W-matched	19	10	1	6	4	17	5	2	11	18	20	8	3							
Row 2	M-matched	5	9	10	17	15	3	6	13	14	4	1	11	2							
Row 3	Risk levels	3	3	4	5	6	6	6	6	6	6	6	6	6							
Row 4	W-payoffs	70 €	70 €	70 €	70 €	70 €	70 €	40 €	70 €	40 €	70 €	70 €	40 €	40 €	40 €	40 €	40 €	70 €	70 €	70 €	70 €
Row 5	M-payoffs	70 €	70 €	70 €	70 €	70 €	70 €	40 €	40 €	70 €	70 €	70 €	40 €	70 €	70 €	70 €	40 €	70 €	40 €	40 €	40 €

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Addendum

We deem that it is necessary to provide a full proof of all propositions.

Proposition 1 Presented in terms of graph theory, the proposition would be obvious. Treating the formation of coalitions as a chain of sets X_k, $\overline{1,k}$, the proposition may be explained in the form of a chain of graphs $C(X_k)$, whereby the lowest risk $F(X_k)$ is assigned to couples α ready to match in the list $\left\langle \alpha = arg\,min_{\sigma \in X_k} \pi(\sigma, X_k) \right\rangle$. The list represents a graph $C(X_k)$ with edges $\left\langle [i,j] = \alpha \right\rangle$. Suppose that a couple $\sigma \in X_k$, not necessarily listed in $C(X_k)$, decides to date. The couple σ leaves the game. As a result, some less risky couples $\alpha \in C(X_k)$ must reconsider whom they prefer to date, as their preferred partners, while the chain X_k is under formation, are no longer available. There are two possibilities. First, all partners, who are yet unmatched and are present in couples $\alpha \in C(X_k)$, preferred at least one of two partners in σ, i.e., all these couples α are adjacent to σ in the graph $C(X_k)$. Second, because for some couples $\alpha' \in C(X_k)$ not adjacent to couple σ, the partners of σ do not appear for α' in the list $C(X_k)$. The proposition presupposed that, in the process of coalitions' X_k formation, the lowest risk function $F(X_k)$ does not decrease. Therefore, the statement of the proposition $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$ holds in both situations.

Proposition 2 The proof is explained in the basic terms. The idea is to apply a mathematical induction scheme. We claim that, starting from the initial state $\mathcal{P}$ of the game, where nobody has been matched yet, it is possible to reach an arbitrary coalition X by sequence $\langle \alpha_1,...,\alpha_k \rangle$, $X_1 = \mathcal{P}$, $X_{k+1} = X_k - \alpha_k$ $X = X_k$, $\overline{1,k}$. The sequence will improve the payoffs x_k previous steps $\langle \alpha_1,...,\alpha_{k-1} \rangle$ in accordance with non-decreasing values $F(X_k)$. First, the statement of the proposition can be verified by observation of all preference tables and all coalitions X that emerged from all $n \times m$ tables, when both n and m are small integers. For higher n and m values, it is NP-hard problem. Second, consider an arbitrary coalition X of the $n \times m$-game. While the coalition $\overline{X} = D_x$ includes all matched couples, in order to arrange a new couple, all participants in X are still unmatched. We can thus always find a couple $\alpha_0 \in \overline{X}$ such that $F(\mathcal{P}) \le F(\mathcal{P} - \alpha_0)$. Consider $(n-1)\times(m-1)$-game, which can be arranged from $n \times m$-game by declaring the partners of the couple α_0 as dummy players $\delta \notin \mathcal{P}$. By the induction scheme, there exists a sequence of pairs $\langle \alpha_1,...,\alpha_k \rangle$ with required quality of improving the payoffs x_k starting from $X_1 = \mathcal{P} - \alpha_0$. Restoring the dummy couple α_0 to the role of players in the $n \times m$-game, we can, in particular, ensure the required quality of the sequence $\langle \alpha_0, \alpha_1,...,\alpha_k \rangle$. The statement of the proposition is obviously the corollary of the claim above. However, it is clear that, ensured by its logic, the claim is a more general statement than the statement of the proposition.

Proposition 3 The first part of the statement is self-explanatory. The coalition $\mathcal{N}$ stops being a proper subset among kernels $\{\mathcal{K}\}$ as soon as the payoff function $F(\mathcal{N})$ allows improving the outcome $\boldsymbol{n}$. The second part of the proposition is the same statement, worded differently.