Quasilinear Monotone Systems *

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A class of monotone systems, called quasilinear, is introduced. It is shown that an arbitrary monotone system can be represented as a combination of such systems. It is established that the construction of monotone system, defined on Boolean, can be carried out to an arbitrary finite distributive sublattice.

1. Introduction

Monotone systems have been introduced by Mullat [1] as a method of modeling and analysis of composite systems of interconnected elements. An attempt of their use [2] has shown the importance of the construction of a universal mechanism for the generation of such systems. In this connection, in [3] one has suggested a heuristic method for the construction of a large collection of various systems.

It is known [4] that the macro-description of a monotone system, i.e., the collection of the embedded families of level sets, is a subsemilattice of sets. The possibility of such a lattice macro-description makes this method workable in applied investigations, where the fundamental aim is to give a qualitative characteristic for a composite system.

In this paper we generalize the method of monotone systems. Monotone systems are constructed not only on semilattices of sets but also on finite distributive semilattices, while some of their elements are also on arbitrary semilattices. This, in turn, allows us to pose and solve problems of search of informative attributes for the construction of recognition rules and regression equations [5], and also the analysis of sublattices of partitions, generated by hierarchies, in particular of results of arbitrary agglomerative classification procedures [6].

Below we introduce also a class of monotone systems, called quasilinear, which is remarkable by the fact that an arbitrary monotone system is constructed in the standard manner from the systems of the same class. This, one detects a fundamental possibility to select for each concrete case the most adequate system. It is interesting to mention that this class is entirely determined by a special family of matrices, arising in game theory.

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The separation of quasilinear systems allows us to represent intuitively the structure of the set of such macro-description of all possible monotone systems. This representation turns out to be a convenient means for the separation of classes of monotone systems with given macro-properties.

In Appendix I we give a summary of the necessary definitions and facts from the theory of lattices, while in Appendix II we give the proofs of all the theorems of the subsequent sections.

2. Generalized Monotone Systems and Quasilinearity

In [7] it has been proven that the problem of the separation of the characteristic subsets (kernels) of monotone systems can be solved not only on the entire Boolean 2^W (W is a finite set, $|W| = N$), but also on any of its subsemilattices. However, this generalization carries a serious limitation. For example, with its aid one cannot describe the interdependencies defined on the set of partitions. A subsequent generalization is connected with the investigation of the possibility of the construction of monotone systems on arbitrary finite semilattices.

The obtained results turn out to be important also for a deeper understanding of the known construction of monotone systems on a Boolean. Therefore, in the presentation, the general facts will be complemented by their illustration in the indicated special case.

Definition 1. A function $F: L \to \Re$ on semilattice is said to be a quasiconcave (quasiconvex) if $\forall x, y \in L$

$$F(x \vee y) \geq \min\left(F(x), F(y)\right), \tag{1}$$

$$F(x \vee y) \leq \max\left(F(x), F(y)\right). \tag{2}$$

A function, satisfying simultaneously (1) and (2), is said to be a quasilinear.

Example 1. L is a set of vectors $x = \left\{x^1, \dots, x^n\right\}$ in $\Re^n$, closed with respect to the operation $x \vee y = \left\{\max\left\{x^1, y^1\right\}, \dots, \max\left\{x^n, y^n\right\}\right\}$. The function $\min\left(x^j \,\middle|\, j = \overline{1,n}\right)$ is quasiconcave and, under the additional conditions $x^1 \geq x^2 \geq \dots \geq x^n$, it is quasilinear.

Example 2. A monotone system on a Boolean [1] [1-4].

A monotone system is defined as a collection $\langle W,\pi,F\rangle$, where W is a finite set, $|W| = N$, $\pi(i,H)$ is a function on $W \times 2^W$, defined on all pairs (i,H), where $i \in H$. Regarding $\pi(i,H)$, we assume that it is monotone with respect to the second argument:

$$\pi(i,H \setminus \{k\}) \le \pi(i,H) \ \ \forall\, i,k \in H \subseteq W,$$

while the function $F(H)$ is defined in terms of $\pi(i,H)$:

$$F(H) = \min_{i \in H} \pi(i,H).$$

It is known [8] that $F(H)$ satisfies the quasiconcavity condition.

Let L be a finite semilattice, let $J(L)$ be the set of its indecomposable elements (without zero), and let $J(x)$ be the subset of those elements from $J(L)$, which are smaller than or equal to x.

In this section we consider functions defined on the semilattice $L^0 = L \setminus \{0\}$, where 0 is the zero of the semilattice L. This will not be mentioned specially in the sequel and when and when we talk of functions, defined on L, then we assume that their values are not defined on the element 0. We note that the choice of an arbitrary value $F(0)$ does not destroy the inequalities (1) and (2).

Theorem 1. *In order that a function $F(x)$ defined on a finite distributive semilattice L be quasiconcave, it is necessary and sufficient that there exist a function $\pi^0 : J(L) \to \Re$, which is monotone with respect to the second argument,*

$$\forall\, a \in J(L),\ \forall\, \ell,\ell' \in L : a \le \ell \le \ell' \Rightarrow \pi^0(a,\ell) \le \pi^0(a,\ell') \tag{3}$$

and for which

$$\forall\, x \in L^0 : F(x) = \min_{a \in J(x)} \pi^0(a,\ell'). \tag{4}$$

Moreover,

$$\pi^0(a,x) = \max_{y \in [a,x]} F(y), \tag{5}$$

where $[a,x] = \{\ell \in L \mid a \le \ell \le x\}$.

This theorem is a generalization of a fact, proved in [8], for monotone systems on a Boolean.

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[1] We note that a Boolean can be considered, in particular, as a semilattice with respect to the operation $\cup$.

The selection of the distributive semilattices, for the investigation of the generation of monotone systems, is not accidental. As shown by the next theorem, we cannot relax the distributivity property and preserve the fundamental relations (3)-(5).

Theorem 2. *If each quasiconcave function $F(x)$, defined on a finite semilattice L, containing with any two elements also their infimum, can be represented in the form (4), where $\pi^\theta(a,\ell)$, satisfying (3), is expressed in terms of $F(x)$ with the aid of (5), then L is distributive.* [2]

With the aid of $F(x)$, defined on a finite semilattice L, we construct a function of two variables $g(a,b)=F(a\vee b)$, where $a,b\in J(L)$, i.e. g is defined on the set $J(L)\times J(L)$.

Theorem 3. *If F is a quasilinear function on an arbitrary finite semilattice L, then F can be represented in the form*

$$F(x)=\min_{a\in J(x)}\max_{b\in J(x)} g(a,b)=\max_{a\in J(x)}\min_{b\in J(x)} g(a,b). \tag{6}$$

If L is distributive, then each function of the form (6) is quasilinear.

The monotone systems, generated by quasilinear functions, belong, in addition, to a certain special class.

Definition 1. By an increment of a monotone system $\pi^\theta(a,\ell)$ on a triple $a,\ell,\ell'\in L$, $a\le\ell\le\ell'$, $a\in J(L)$, we mean $\Pi(a,\ell,\ell')=\pi^\theta(a,\ell')-\pi^\theta(a,\ell)$.

Definition 2. A monotone function $\pi(a,l)$ has an antimonotone increment if $\forall a\in J(L)$, $\forall\,\ell,\ell',\ell_\theta,\ell'_\theta$, $\ell\le\ell'$, $a\le\ell_\theta\le\ell$, $a\le\ell'_\theta\le\ell'$ and $J(\ell')-J(\ell)=J(\ell'_\theta)-J(\ell_\theta)$ we have $\Pi(a,\ell,\ell')\le\Pi(a,\ell_\theta,\ell'_\theta)$.

Theorem 4. *On a finite distributive semilattice L, a monotone function $\pi^\theta(a,\ell)$, generated by a quasilinear function according to the relation (5) has antimonotone increments.*

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[2] We have in mind a semilattice, containing with any two elements also their infimum, and not a lattice, since if $L\subseteq L'$, where L' is a lattice (for example, 2^W), then the infimum of $x,y\in L$ is not necessarily equal to $x\wedge y$ (in particular, to $X\cap Y$, if $L'=2^W$).

Monotone systems with antimonotone increments have been considered in [9], where the concept of their local transformations is investigated. In the case of monotone systems on a Boolean, this theorem establishes the quasilinearity of certain functions, arising in the theory of zero-sum two-person games.

Definition 3. A matrix $A = \left\| a_{ij} \right\|_N^N$, $i, j \in W$, is said to be stable if it is symmetric and $\forall H \subseteq W$ it has a saddle point

$$\min_{i \in H} \max_{j \in H} a_{ij} = \max_{i \in H} \min_{j \in H} a_{ij} . \qquad (7)$$

Definition 4. A monotone system $\langle W, \pi^0, F \rangle$ is said to be a quasilinear if F is quasilinear function.

Corollary of Theorem 3. A monotone system $\langle W, \pi^0, F \rangle$ is quasilinear if and only if the functions π^0 and F can be represented in the form

$$\pi^0(i, H) = \max_{j \in H} a_{ij} , \qquad (8)$$

$$F(H) = \min_{i \in H} \max_{j \in H} a_{ij} , \qquad (9)$$

where $A = \left\| a_{ij} \right\|$ is a stable matrix.

Theorem 5. *In order that a matrix* A *be stable it is sufficient that* $\forall\, i, j, k, l : [a_{ik}, a_{jl}] \cap [a_{il}, a_{jk}] \neq \varnothing$.

Theorem 5 is a particular case of Shapley's theorem [10], which asserts that the indicated conditions are sufficient that the payoff matrix of a zero-sum two-person game should have a saddle point.

Example 3. Assume that the matrix $A = \left\| a_{ij} \right\|$ is such that $a_{ij} = \max\left(a_{ii}, a_{jj}\right)$ and the collection $\{a_{11}, \ldots, a_{NN}\}$ of diagonal elements is arbitrary. Then the matrix A is stable and the function $F(H) = \max_{i \in H} a_{ii}$ is quasilinear.

Example 4. We consider an undirected graph G, for which W is the set of its vertices. By E we denote the set of its edges. Assume that it has the following two properties:

a) the set W is partitioned into two subsets H^* and $W \setminus H^*$, such that $\forall i,j \in H^*: (i,j) \in E$; $\forall i,j \in W \setminus H^*: (i,j) \notin E$;

b) G does not contain four-vertex subgraphs [3] of the form given in Fig. 1.

Fig. 1

It is easy to show that 0.1-stable matrices are the adjacency matrices of such graphs.

The problem of the description of the semilattice of all kernels of monotone systems, i.e., the determination of the systems generators of the semilattice of those $H^* \subseteq W$, such that $F(H^*) = \max\left(F(H) \middle| H \subseteq W\right)$, where F is a quasiconcave function, is sufficiently complex. Presently, no polynomial algorithm is known for its solution in the general form. However, for the description of the semilattices of the kernels of a quasilinear monotone system on can give a polynomial algorithm.

Let $\langle W,\pi,F \rangle$ be a quasilinear monotone system. We construct the matrix $A = \|a_{ij}\|$ from the function F in the above-described manner. From this matrix, in accordance with the corollary to Theorem 3, we construct a monotone system $\pi_1^0(i,H) = \max_{j \in H} a_{ij}$ and its dual $\pi_2^0(i,H) = \min_{j \in H} a_{ij}$. Let $i_{H*} \in H$ be an element on which the value

$$F(H) = \pi_1^0(i_{H*},H) = \pi_2^0(i_{H*},H).$$

Theorem 6. *If H^0 is a kernel of the monotone system $\langle W,\pi_1^0,F \rangle$, then also any $H \in [i_{H^0*},H^0]$ is its kernel. An algorithm for the determination of the rights and left endpoints of the intervals $[i_{H^0*},H^0]$ consists in the successive determination of the elements i_{H^0*} with the aid of the procedure of the separation of a maximal kernel of the monotone system and the passage to the subset $H^0 \setminus \{i_{H^0*}\}$.*

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[3] A subgraph of a graph is an arbitrary subset of vertices from W, together with edges, for each of which both incident vertices belong to the given selected subset of vertices.

Everything that has been said is valid also for monotone systems on an arbitrary distributive semilattice.

On nondistributive semilattices it is easy to construct examples of functions which cab be represented in the form (6), but are not quasilinear. These are, for example, the functions $f : M_5 \to \Re$ and $g : N_5 \to \Re$, where $f(0) = 0$, $f(a) = 1$, $f(b) = 3$, $F(c) = 4$, $f(I) = 2$; $g(0) = 0$, $g(a) = 4$, $g(b) = 1$, $g(c) = 3$, $g(I) = 2$.

Theorems 1-3 show that the distributivity property of semilattice allows the construction on it of an exact analog of monotone systems on a Boolean. On the other hand, the possibility of the construction of such an analog appears as a distinctive characterization of distributivity.

3. Convex Analysis on Semilattices

The set of the subsemilattices of an arbitrary semilattice forms a convexity in the sense of axiomatic definition [11]. The quasiconvex functions form the basis of convex analysis on semilattices.

Definition 5. A subsemilattice L' of an arbitrary semilattice L is said to be separating if $L - L'$ is a semilattice.

Example 5. Assume that L, as in Example 1, is a set of vectors, closed with respect to $\vee$, and assume that $L' \subset L$ is defined by the system of inequalities $0 \le x^i \le b^i$, $i = \overline{1,n}$. Then L' is separating semilattice.

The following theorems show that the separating semilattices, in the problem of optimization of quasiconcave functions of sets on semilattices, play the role of semispaces from the convex analysis of functions on the Euclidean space.

Theorem 7. *For an arbitrary semilattice L and any of its nonintersection subsemilattices L_1 and L_2 ($L_1 \cap L_2 = \varnothing$) there exists a separating semilattice L' such that $L_1 \subseteq L'$, $L_2 \subseteq L - L'$.*

Theorem 8. *An arbitrary subsemilattice of an arbitrary semilattice can be represented in the form of the intersection of separating subsemilattices*.

Theorem 9. *The subsemilattices of an arbitrary semilattice, and only them, can be represented in the form of level sets*

$$L_u = \left\{ x \in L \middle| F(x) \geq u \right\}$$

of quasiconcave functions. The separating subsemilattices of an arbitrary semilattice, and only them, can be represented in the form of level sets of quasilinear functions.

Corollary 1. Let L_1 and L_2 be two nonintersecting subsemilattices of semilattice L and assume that some function F takes the value u_1 on the elements of L_1 and the value u_2 on the elements of L_2. Then F can be extended to a quasilinear function to the entire semilattice L.

Corollary 2. Let $\left\langle W, \pi^1, F_1 \right\rangle$, $\left\langle W, \pi^2, F_2 \right\rangle$ be two monotone systems such that the corresponding families of subsets, on which the functions F_1 and F_2 assume a maximal value (kernels), do not intersect. Then one can construct a quasilinear function F such that all the kernels of F_1 are the kernels of F and all kernels of F_2 are kernels of $\left(-F\right)$.

Theorem 10. *On any semilattice the quasiconcave functions, and only them, can be represented in the form*

$$F = \min_{k \in K} F_k \,, \tag{10}$$

where all F_k are quasilinear functions, while K is some set of indices.

Theorems 7-9 are valid also for infinite semilattices.

Theorem 11. *The subsemilattices of a distributive semilattice L, and only them, can be represented in the form*

$$L - \bigcup_{i \in I} [\, a_i, \ell_i \,] \quad \forall\, i \in I : a_i \in J(\, \ell_i \,) \cup \{0\}; \tag{11}$$

here I is some set of indices.

Theorem 12. *The separating subsemilattices of a distributive semilattice, and only them, can be represented in the form*

$$L - \bigcup_{i \in I} [\, a_i, \ell_i \,] \,, \tag{12}$$

where $\left\{ \ell_i \middle| i \in I \right\}$ is a chain and $\forall\, i \in I : a_i \in J(\, \ell_i \,) \cup \{0\}$.

Theorem 11 is the analog of Rival’s theorem [12] on the structure of the sublattices of a distributive lattice.

We introduce an elementary quasilinear function on a semilattice $(a \in J(l))$:

$$F(x) = \begin{cases} u_1, \ x \in [a, \ell], \\ u_2, \ x \notin [a, \ell]; u_2 > u_1. \end{cases} \tag{13}$$

Corollary (to Theorem 10). Every quasiconcave function on a distributive semilattice is the minimum of elementary quasilinear functions.

Corollary (to Theorem 12). Every co-indecomposable element of the lattice of subsemilattices of a distributive semilattice L is the level set of some elementary quasilinear function.

Example 6. We consider special quasilinear characteristic functions on 2^W, defined by separating semilattices of the form

$$L = 2^W - [\{i\}, W \setminus \{j\}].$$

They can be computed by the formula

$$F_{ij}(H) = \begin{cases} -1, & \text{if } H \in [\{i\}, W \setminus \{j\}], \\ 1 & \text{otherwise.} \end{cases}$$

We construct the function

$$F(H) = min\left(c_{ij} \cdot F_{ij}(H) \,\middle|\, i, j \in W, i \neq j\right), \tag{14}$$

where $\| c_{ij} \|$ is an arbitrary matrix $(c_{ij} \geq 0)$. It is easy to show that the function (14) is quasi-flowlike [8] and, therefore, it can be written in the form $F(H) = \max\limits_{i \in H} \max\limits_{j \in W \setminus H} c_{ij}$. Such functions are important from the point of view of applications since with their aid one models the partition of an analyzed set [13].

We say that a semilattice $L' \subseteq L$ is dense if $\forall x, y \in L'$ such that $x < y$ there exists a maximal chain in L such that $x = x_1 < x_2 < ... < x_k = y$, $x_1, ..., x_k \in L'$.

A semilattice L is said to be interval-nonclosed if no nonempty open interval in it is a semilattice, i.e.,

$$\forall\, x < y\; \exists\, a, b \in [\, x, y\,] \setminus \{x, y\} : a \vee b = y \,.$$

Theorem 13. *In order that each separating subsemilattice L' be dense in L, it is necessary and sufficient that L be interval-nonclosed.*

Obviously, a Boolean is an interval-nonclosed semilattice. It is easy to show that the property of being interval-nonclosed is possessed also by partition lattices and by geometric lattices (this follows from the existence of relative complements in the lattices of the mentioned types [14]).

The separating semilattices of a Boolean are dense also in a certain stronger sense.

Theorem 14. *For any separating semilattice of sets we have: $\forall\, H_1, H_2 \in L$ there exists a sequence of sets $\{Q_1, \ldots, Q_k\}$, $Q_i \in L$, $i = \overline{1,k}$, such that $H_1 = Q_1$, $H_2 = Q_k$ and* $\big|\, |Q_{i+1}| - |Q_i| \,\big| = 1$, $i = \overline{1, k-1}$.

Appendix 1. Summary of the necessary results from lattice theory

<u>Definition 1</u> [14]. A set L, with a binary operation defined on it, is said to be a semilattice if $\forall\, x, y \in L : x \vee y \in L$ and the following properties hold:

1) idempotence: $A\, x \in L : x \vee x = x$,

2) commutativity: $\forall\, x, y \in L : x \vee y = y \vee x$,

3) associativity: $\forall\, x, y, z \in L : x \vee (y \vee z) = (x \vee y) \vee z$.

<u>Definition 2</u> [14]. A set L, with two binary operations $\vee$ and $\wedge$ defined on it, is said to be a lattice if $\forall\, x, y \in L : x \vee y \in L, x \wedge y \in L$, the operations $\wedge$ and $\vee$ are idempotent, commutative, associative, and the absorption laws are satisfies: $\forall\, x, y \in L : x \vee (x \wedge y) = x$; $x \wedge (x \vee y) = x$.

Definition 3 [14]. A lattice is said to be a distributive if $\forall x,y,z \in L$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 4 [14]. A partial order on a semilattice (and on a lattice) is introduced in the following manner: $x \le y \Leftrightarrow x \vee y = y$.

Definition 5. By a chain in a lattice (semilattice) L we mean an ordered set in which any two elements are comparable. A chain in a finite lattice (semilattice) is said to be maximal if each of its elements is covered by the following one: $a \succ b \Leftrightarrow a \ge b \,\&\, (a > x \ge b \Rightarrow x = b)$.

Definition 6 [14,15]. A semilattice L is said to be distributive if $\forall x,y,z \in L$, such that $z \le x \vee y$, there exists $a,b \in L$, possessing the property that $a \le x$, $b \le y$, $a \vee b = z$.

The importance of the construction of distributive lattices and semilattices is due to the following circumstance. The distributive lattices, and only they, are isomorphic to rings of sets, i.e., for distributive lattice L there exists W (finite if L is finite) and a correspondence $x \to H(x) \subseteq W$, such that $x = y \Leftrightarrow H(x) = H(y)$ and $H(x \vee y) = H(x) \cup H(y)$, $H(x \wedge y) = H(x) \cap H(y)$ [14]. In other words, the operations $\vee$ and $\wedge$ are the analogs of $\cup$ and $\cap$. The distributive semilattices, and only they, are isomorphic to semirings of sets, i.e., the above-introduced mapping $x \to H(x)$ does not have the last of the above-enumerated properties and if $x,y \in L$ there exists a largest $z \in L$, such that $z \le x$ and $z \le y$, then $H(z) = H(x) \cap H(y)$ [16].

There exists another distinctive particularity of distributive lattices and semilattices. A lattice is distributive if and only if for any two of its elements x and y there exists a largest z such that $z \le x$ and $z \le y$, and it does not contain as a retract either N_5 or M_5 [4] [15] (see fig.2).

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[4] A semilattice L_1 contains as a retract – a semilattice L_2 if there exists mapping $f_1 : L_1 \to L_2$ and $f_2 : L_2 \to L_1$, preserving the operation $\vee$, i.e. $f_i(x \vee y) = f_i(x) \vee f_i(y)$, $i = 1,2$, such that $f_1 \circ f_2$ is the identity mapping.

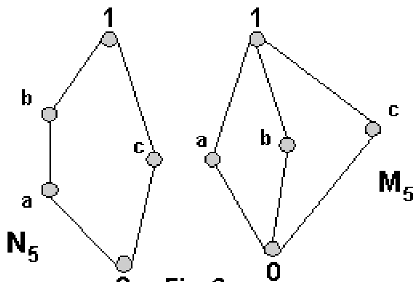

Fig. 2

Definition 7 [14]. An element x of a semilattice L is said to be irreducible if from $x = a \vee b$, $a,b \in L$ there follows $x = a$ or $x = b$. An element 0 is called the zero of a semilattice if $0 \le x$ for all $x \in L$. [5] The set of all irreducible elements without 0 will be denoted by $J(L)$. In addition, $J(x) = \{a \in J(L) \mid a \le x\}$.

Definition 8 [14]. By the complement $\bar{x}$ of an element x of a semilattice L we mean its complement with respect to 1: $x \vee \bar{x} = 1$. The element 1 is called the identity of a semilattice if $1 \ge x$ for all $x \in L$. An element x is said to be meet-irreducible if $x = a \wedge b$, $a,b \in L$ imply $x = a$ or $x = b$.

Definition 9 [14]. A lattice R has relative components if $\forall x \le y$, $\forall z \in [x,y]$, $\exists z^1 \in [x,y] : z \vee z^1 = y$, $z \wedge z^1 = x$.

Definition 10 [14]. By a $\vee$-homomorphism we mean a mapping of a semilattice into a semilattice, preserving the operation $\vee$.

Appendix 2

The proof of Theorem 1 is similar to the proof of the corresponding theorem of [8] since for distributive semilattices we have: $\forall x,y \in L : J(x \vee y) = J(x) \vee J(y)$.

Proof of Theorem 2. We precede the proof by two lemmas.

[5] A distributive semilattice always contains 0 [14,15].

LEMMA 1. Let L be a semilattice containing N_5 as a retract. Then there exists $x\in J(L)$, such that $x\in J(b)$, but $x\notin J(a)\cup J(c)$.

Proof. Let $f_1 : L\rightarrow N_5$, $f_2 : N_5\rightarrow L$ be $\vee$- homomorphism and let $f_1\circ f_2$ be the identity mapping. We identify N_5 with its image $f_2(N_5)\subseteq L$; then f_1 is identified with the mapping $\varphi : L\rightarrow N_5$, where φ is a $\vee$- homomorphism and the restriction of φ to $N_5\subseteq L$ is the identity. Let $x\in N_5$, $y\le x$. Since $x=\varphi(x)=\varphi(x\vee y)=\varphi(x)\vee\varphi(y)=x\vee\varphi(y)$, we have $\varphi(y)\le x$. Therefor, $\varphi(\vee y \mid y\in J(x))=x$, for $x\in N_5$. We consider $x=\varphi(\vee y \mid y\in J(b)-J(a))$. Clearly, $x\le b$. If $x<b$, then, due to the fact that $\forall\, y\in J(a):\varphi(y)<a$, we have $\varphi(\vee y \mid y\in J(b)<b)$, which is not possible. Thus, $x=b$.

Assume that the assertion of the lemma is false. Then $J(b)-J(a)\subseteq J(c)$, i.e., $b=x\le\varphi(\vee y \mid y\in J(c))=c$, which is not true. The obtained contradiction proves the lemma. ■

LEMMA 2. Let L be a semilattice containing M_5 as a retract. Then there exists $x\in J(L)$, such that $x\in J(b)$, but $x\notin J(a)\cup J(c)$.

The proof is similar to that of the previous lemma.

Assume now that L is a semilattice, containing for any two elements their greatest lower bound, and that L is not distributive. Then L contains N_5 or M_5 as a retract. Let x^* be the element, whose existence is asserted in lemmas 1 and 2. We set $\forall\, x\ge x^*$, $\forall\, p\in J(y):\pi(x^*,x)<\pi(p,y)$ and the function π is monotone. We construct a function F according to the rule (4). We obtain: $F(I)\le\pi(x^*,I)$, $F(a),F(c)>\pi(x^*,I)$, since $x^*\notin J(a)\cup J(c)$. Thus, $F(I)=F(a\vee c)<\min(F(a),F(c))$, i.e., F is not quasiconcave. The theorem is proved. ■

Proof of Theorem 3. We show that a quasilinear $F(x)$ can be represented in the form

$$F(x)=\min_{a\in J(x)}\max_{b\in J(x)} g(a,b), \tag{A.1}$$

where $g(a,b) = F(a \vee b)$. Let b_a be an element of $J(x)$ on which the maximum of $g(a,b)$ is attained. We denote the right-hand side of (A.1) by $R(x)$. Then, in accordance with (1), we have $R(x) = \min F(a \vee b_a) \le F(x)$. Let $R(x) < F(x)$. In this case there exists $a \in J(x)$ such that $\forall b \in J(x): F(a \vee b) < F(x)$, and therefore, on the basis of (2) we have $F(x) \le \max \left(F(a \vee b) \middle| b \in J(x) < F(x) \right.$. The obtained contradiction proves that $R = F$. In similar manner one proves the representation of F in the form of a maximin function.

We prove that if L is distributive, then a function of the form (6) is quasilinear. The function $\pi_1(a,x) = \max\limits_{b \in J(x)} g(a,b)$ is monotone with respect to x, $a \in J(x)$. By virtue of $J(x \vee y) = J(x) \cup J(y)$, we have $a \in J(x \vee y) \Rightarrow a \in J(x) \vee a \in J(y)$.

Assume, for the sake of definiteness, that
$\arg F(x \vee y) = \arg\min \pi_1(a, x \vee y) = a_1 \in J(x)$ and let
$F(x) = \pi_1(a_2, x) \le \pi_1(a_1, x) \le \pi_1(a_1, x \vee y) = F(x \vee y)$. Thus, the function $F(x)$ is quasi-concave. Similarly, from the fact that $\pi_2(a,x) = \min\limits_{b \in J(x)} g(a,b)$ is antimonotone with respect to x, there follows that quasiconvexity of the function $F(x)$. The theorem is proved. ■

<u>Proof of Theorem 4</u>. We consider the representation (6) of a quasilinear function. In accordance with this representation, we have $\pi^0(a,\ell) = \max g(a,b)$, $b \in J(\ell)$. We note that $\Pi(a,x,x') \ge 0 \ \ \forall a \in J(\ell)$, $x,x' \in L$, $a \le x \le x'$.

Let $\Pi(a,l,l') = \max\limits_{b \in J(\ell)} g(a,b) - \max\limits_{b \in J(\ell')} g(a,b) > 0$. This means that

$$\begin{aligned}
&b^* = \arg\max \left(g(a,b) \middle| b \in J(\ell) \right) \in J(\ell') - J(\ell) = J(\ell'_0) - J(\ell_0), \text{i.e.,} \\
&b^* = \arg\max \left(g(a,b) \middle| b \in J(\ell'_0) \right), \\
&\pi^0(a,\ell') = \pi^0(a,\ell'_0).
\end{aligned} \tag{A.2}$$

At the same time, from the monotonicity there follows that

$$\pi^0(a,\ell) \ge \pi^0(a,\ell_0). \tag{A.3}$$

From (A.2) and (A.3) we have

$$\Pi(a,\ell,\ell') \le \Pi(a,\ell_0,\ell'_0).$$

In the case $\Pi(a,\ell,\ell') = 0$, the validity of the required condition is obvious. The theorem is proved. ■

Proof of Theorem 5. By Shapley's theorem [10], for the existence of a saddle point for a matrix it is sufficient that each of its 2×2 submatrices should have a saddle point. Now, Theorem 5 follows from the following lemma, proved by straightforward verification.

LEMMA 3. In order that 2×2 matrix A should have a saddle point, it is necessary and sufficient that $[a_{11},a_{22}]\cap[a_{12},a_{21}]\neq\varnothing$.

Proof of Theorem 6. In accordance with the definition of a kernel and the corollary to Theorem 3, we have

$$\max_{H\subseteq W} F(H)=F(H^0)=\pi_2^0(i^*_{H^0},H^0)=\max_{i\in H^0}\pi_2^0(i,H^0).$$

Let $i^*_{H^0}\in H\subseteq H^0$. By virtue of the antimonotonicity of the function π_2^0:

$$F(H)=\max_{i\in H}\pi_2^0(i,H)\geq\pi_2^0(i^*_{H^0},H)\geq\pi_2^0(i^*_{H^0},H^0)=F(H^0),$$

i.e., also H is a kernel. The theorem is proved. ■

Proof of the Theorem 7.

Let $L_0\subseteq L$ and $x\in L-L_0$. Then by $[L_0,x]=L_0\cup\{x\}\cup\{\ell\vee x \mid \ell\in L_0\}$ we denote the smallest subsemilattice L, containing L_0 and x. The proof is based on the following lemma.

LEMMA 4. Let $L_1\cap L_2=\varnothing$, $x\in L-(L_1\cup L_2)$. The either $[L_1,x]\cap L_2=\varnothing$ or $[L_2,x]\cap L_1=\varnothing$.

Proof. We assume the opposite, i.e., $[L_1,x]\cap L_2\neq\varnothing$ and $[L_2,x]\cap L_1\neq\varnothing$. Since $L_1\cap L_2=\varnothing$, there exists $\ell_1\in L_1$, $\ell_2\in L_2$ such that $\ell_1\vee x\in L_2$, $\ell_2\vee x\in L_1$. Let $a=\ell_1\vee\ell_2\vee x$. Since $a=(\ell_1\vee x)\vee\ell_2$, we have $a\in L_2$ and since $a=(\ell_2\vee x)\vee\ell_1$, we have $a\in L_1$, i.e., $a\in L_1\cap L_2$. Thus, we obtained a contradiction. The lemma is proved. ■

We prove the theorem by induction on $k=|L-(L_1\cup L_2)|$. If $k=0$, then the assertion of the theorem is obvious. Assume that the theorem holds for all $k<m$ and we prove that it holds for $k=m$. We consider $x\in L-(L_1\cup L_2)$ and assume, for the sake of definiteness, that $[L_1,x]\cap L_2=\varnothing$. We set $[L_1,x]=L'_1$, $L'_2=L_2$. The cardinality $|L-(L'_1\cup L'_2)|<m$ and, therefore, there exists $L'\subseteq L$ such that $L_1\subseteq L'_1\subseteq L'$ and $L'_2=L_2\subseteq L-L'$. The theorem is proved. ■

Proof of Theorem 8. Let L be a semilattice, let L' be a subsemilattice in it, and let $x \in L'$. Then, by Theorem 7, there exists a separating semilattice L_x such that $L \subseteq L_x$, $x \notin L_x$. The assertion of the theorem follows from the equality

$$L' = \bigcap \left(L_x \mid x \notin L' \right). \tag{A.4}$$

Proof of the Theorem 9. The facts that the level set of a quasiconcave function is a semilattice and that the level set of a quasilinear function is a separating semilattice are sufficiently obvious. The function

$$F_{L_0}(x) = \begin{cases} 1 & \text{for } x \in L_0, \\ 0 & \text{for } x \notin L_0 \end{cases}$$

is quasiconcave on any semilattice L_0 and quasilinear if L_0 is separating. Its level set $\{ x \in L \mid F_{L_0}(x) \geq 1 \}$. Theorem is proved. ■

Proof of Theorem 10. a) Assume that the function $F(x)$ can be represented in the form (10). Then

$$\forall\, x_1, x_2 \in L : F(x_1 \vee x_2) = \min_{k \in K} F_k(x_1 \vee x_2) = F_{k^0}(x_1 \vee x_2) \geq$$
$$\geq \min \left(F_{k^0}(x_1), F_{k^0}(x_2) \right) \geq \min \left(F(x_1, F(x_2) \right), \text{ i.e.,}$$

the function $F(x)$ is quasiconcave;

b) Let F be a quasiconcave function on a finite semilattice of L. Since L is finite, it follows that F can take a finite set of values $u_1 < u_2 < ... < u_s$. [@]

We consider the semilattices $L_p = \{ x \in L \mid F(x) > u_p \}$, $p = 1, ..., s-1$. By Theorem 8, we represent each L_p as intersection of separating semilattices: $L_p = \bigcap \left(L_{pk} \mid k \in J_p \right)$.

For each $k \in J_p$ we introduce the function

$$\varphi_p^k(x) = \begin{cases} u_p, & x \notin L_{pk}, \\ u_s, & x \in L_{pk}. \end{cases} \tag{A.5}$$

[@] In Boolean case of monotonic game, see http://www.datalaundering.com/download/monogame.pdf, this series was called a spectrum.

The functions (A.5) are quasilinear. This follows from the fact that if $x_1, x_2 \in L_{pk}$ or $x_1, x_2 \notin L_{pk}$, then $\varphi_p^k(x_1 \vee x_2) = \varphi_p^k(x_i)$, $i = 1,2$. We consider the function

$$r(x) = \min_{p=1,\dots,s-1} \min_{k \in J_p} \varphi_{pk}(x). \tag{A.6}$$

Let $x \in L$ and $F(x) = u_q$, $q < s$, i.e., $x \notin L_q$. Consequently, $x \notin L_{qk}$ for some k and $\varphi_q^k(x) = u_q$. Thus, by virtue of (A.6) we have $r(x) \le u_q$. We assume that $r(x) < F(x)$. Then there exists $\varphi_m^k(x) = u_m$, $m < q$. But this means that $x \notin L_{mk}$ and, therefore, also $x \notin L_m$, i.e., $F(x) \le u_m \le u_q$. The obtained contradiction proves that $r(x) = F(x)$.

The case $q = s$ is considered in similar manner. Relation (A.6) is the desired expansion (10) for the quasiconcave function. The theorem is proved. ■

Proof of Theorem 11. We show that a set of the form (11) is a semilattice. Let $x, y \in L' = L - \bigcup [a_i, \ell_i]$ and let $x \vee y \notin L'$. Then $x \vee y \in [a_i, \ell_i]$ for some i. Since $a_i \le x \vee y$, by the distributivity property we have $a_i = x' \vee y'$, where $x' \le x$, $y' \le y$. But $a_i \in J(L)$, from where $a_i = x'$ or $a_i = y'$, i.e., either $a_i \le x$ or $a_i \le y$. This means that either $x \notin L'$ or $y \notin L'$, since $x, y \le l_i$. Thus, (11) is a semilattice.

Assume that $L' \le L$ is a semilattice and $\ell \notin L'$. We prove that there exists $a \in J(L)$, such that $[a, \ell] \cap L' = \varnothing$. Indeed, suppose this is not so. Then $\forall\, a \in J(\ell)\ \exists\, \ell_a \in [a, \ell] : \ell_a \in L'$. We consider $\ell' = \ell \vee \ell_a$. Since all $\ell_a \in L'$, we have $\ell' \in L'$, but on the other hand, $\ell' \ge \vee(a \in J(\ell))$ and $\ell' \le \ell$, i.e., $\ell' = \ell \notin L'$, which is contradiction. For each $\ell \notin L'$ we denote by $I(\ell)$ the set of $a \in J(L) \cup \{0\}$, such that $[a, \ell] \cap L' = \varnothing$.

We consider a semilattice of the form (11)

$$L'' = L - \bigcup_{\ell \in L} \bigcup_{a \in I(\ell)} [a, \ell]. \tag{A.7}$$

If $x \notin L''$, then $x \in [a, \ell] \cap L' = \varnothing$, $x \notin L'$. If $x \notin L'$, then in accordance with (A.7) we have $x \notin L''$. Thus, $L' = L''$. The theorem is proved. ■

Proof of Theorem 12. We show that (12) is a separating semilattice. It is sufficient to show that $\bigcup [a_i,\ell_i]$ is a semilattice. Let $x_1 \in [a_1,\ell_1]$, $x_2 \in [a_2,\ell_2]$ and $\ell_1 \le \ell_2$. Then $x_1 \vee x_2 \le \ell_2$, $x_1 \vee x_2 \ge a_2$, i.e., $x_1 \vee x_2 \in [a_2,\ell_2]$, which is what we intended to prove.

Conversely, let L_0 be a separating semilattice. We represent it in the form (11). Since L_0 is separating, for $\forall\, p_1,p_2$, $\forall\, x_1 \in [a_{p_1},\ell_{p_1}]$, $\forall\, x_2 \in [a_{p_2},\ell_{p_2}]$ we must have $x_1 \vee x_2 \notin L_0$. From here it follows that the interval $[a_{p_1} \vee a_{p_2},\ell_{p_1} \vee \ell_{p_2}]$ must be contained in $L - L_0$. Making use once again of the fact that L_0 is separating, we obtain: either $[a_{p_1},\ell_{p_1} \vee \ell_{p_2}]$ or $[a_{p_2},\ell_{p_1} \vee \ell_{p_2}]$ is a subset of $L - L_0$. Removing from the expansion (11) the intervals that are entirely included in others, we obtain that for any p_1,p_2 the elements ℓ_{p_1} and ℓ_{p_2} are comparable, i.e., $\{\ell_i \mid i \in I\}$ is a chain. The theorem is proved. ■

Proof of Theorem 13. a) Necessity. Assume that L is not interval-nonclosed, i.e., there exists an open interval $(x,y) = [x,y] \setminus \{x,y\}$ is a semilattice. Also the set $\{x,y\}$ is a semilattice and, therefore, there exists a separating $L' \subseteq L$ such that $(x,y) \subseteq L'$ and $x,y \notin L'$, where $L - L'$ is also separating. In a maximal chain $\{x_i \mid i = \overline{1,k}\}$ such that $x_1 = x$, $x_k = y$ we can have only the elements of $[x,y]$, but $(x,y) \cap L - L' = \varnothing$, i.e., $L - L'$ is not dense.

b) Sufficiency. Let L be interval-nonclosed, $x,y \in L'$ and $x < y$, where L' is a separating subsemilattice of L. We show that for them there exists a connecting sequence. We extend $\{x,y\}$ to a maximal chain in L'. Let a,b be two adjacent elements of this chain, $a \le b$, $a,b \in L'$. If $(a,b) = \varnothing$, then $b \succ a$. Let $(a,b) \ne \varnothing$ and $c \in (a,b)$. Since the chain is maximal, we have $c \notin L'$. By virtue of the interval-nonclosed property, $\exists\, d \in (a,b) : d \vee c = b$. Since $d \in (a,b)$, we have $d \notin L'$ and, therefore, also $d \vee c = b \notin L'$, which is a contradiction. Thus, the connecting sequence has been constructed. If x,y are incomparable, then it is sufficient to consider two connecting sequences: from x to $x \vee y$ and from y to $x \vee y$, since $x \vee y \in L'$. The theorem is proved. ■

Proof of Theorem 14. First we prove the following lemma.

LEMMA 5. Let $L \subseteq 2^W$ be a separating semilattice and let $H, G \in L$, $H \subseteq G$. Then the elements of the set $G \setminus H$ can be ordered, $G \setminus H = \{i_1, ..., i_k\}$, such that all the sets $H \cup \{i_1, ..., i_p\}$, $p \le k$, belong to L.

Proof. (by induction on $k = |G \setminus H|$). If $k = 1$, then the assertion is obvious. Assume that the assertion holds also for $k = m - 1$; we prove it for $k = m$. We show that among the sets $H \cup \{i\}$, $i \in G \setminus H$, at least one belongs to L. Indeed, suppose this is not so. Then $\forall i \in G \setminus H : H \cup \{i\} \notin L$ and, since L is separating, we have $G = \bigcup_{i \in G \setminus H} (H \cup \{i\}) \notin L$, contradiction. We denote by i_1 an element $G \setminus H$ such that $H \cup \{i_1\} \in L$. By the induction hypothesis, $m-1$ elements of $G \setminus (H \cup \{i_1\})$ can be ordered as $\{i_2, ..., i_m\}$, so that all $H \cup \{i_1, ..., i_p\}$, $p = \overline{2, m}$ belong to L. The lemma is proved.

For the proof of the theorem we mention that the set of kernels of a quasilinear function is a separating semilattice (see Theorem 5); we denote it by L. Let $H_1, H_2 \in L$. Then $H = H_1 \cup H_2 \in L$. In accordance with Lemma 5, we order the elements of $H_2 \setminus H_1 : i_1, ..., i_k$ and $H_1 \setminus H_2 : j_1, ..., j_m$. We set $Q_1 = H_1$, $Q_g = H_1 \cup \{i_1, ..., i_{g-1}\}$, $g = \overline{2, k}$; $Q_{k+1} = H$, $Q_{k+g+1} = H \setminus \{j_1, ..., j_{m-g+1}\}$, $g = \overline{1, m}$; $Q_{k+m+1} = H_2$. By virtue of Lemma 4, by construction $\forall g \in \{1, ..., k+m+1\} : Q_g \in L$; $\big| |Q_{g+1}| - |Q_g| \big| = 1$. The theorem is proved. ■

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