The properties affecting ordered functions of sets [1]

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A function (a functional) defined on a set, different from ordinary additive functions (measures), discloses an optimal value

$$F(X) = \max_{x \in X} f(x) \tag{1}$$

of the extremization problem considered as a function of a feasible set.

The function F in (1), as well as its "generator" – the function f, might be non-numeric, but ordered, i.e., to take the values from arbitrary linear ordered set L. However, the ordered functions, produced by the extreme finding problems (1), inherit similar additive properties. Further relaxation of these properties brings about also an exceptional, in similar or distinct sense, classes of set functions, which constitute the subject of current report.

In order to avoid some technical obstacles let restraint it by finite case. Further, once and for all $X \subseteq U$, where U – is fixed finite set. Let F be a certain ordered function on 2^U. We say that F satisfies the ordered additive condition, if

$$F(X' \cup X'') = \max\{F(X'), F(X'')\} \text{ [2]} \tag{2}$$

for each $X', X'' \subseteq U$.

Theorem 1. *To be an ordered additive function for F it is necessary and sufficient that it can be represented in the form of (1). Herein, by the necessity prevails that* $f(x) = F(\{x\})$, $x \in U$.

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[1] An extended thesis's variant of the report at the "*All Union Seminar on Optimization and its Implementations*," Dushanbe, 1986, pp. 151-152.
[2] Later called a quasi-additive condition, trans. remark.

Keeping in mind this theorem, we will call each function arranged by (1) with certain $f(x)$ as ordered additive. Now, it is possible to consider more general case, when the range of F definition not necessarily extends to all $X \subseteq U$, but to certain subfamily $\mathrm{X} \subseteq 2^U$. In particular, F might be not defined on single element subsets. Consider following constraint: for all $X \in \mathrm{X}$ and $\{X_v\} \subseteq \mathrm{X}$, such that $\bigcup_v X_v \supseteq X$, holds

$$F(X) \leq \max_v F(X_v) \ . \qquad (3)$$

Theorem 2. *To be an ordered additive function for* $F(X)$, *defined on some* $\mathrm{X} \subseteq 2^U$, *it is necessary and sufficient the constraint (3) fulfillment.*

Further, for simplicity, let F definition range extends over all proper subsets $X \subset U$.

We call the function $F(X)$ *monotone*, if

$$F(X') \leq F(X'') \text{ by } X' \subseteq X'', \qquad (4)$$

and let call $F(X)$ as $\bigcup$-quasiconvex, if

$$F(X' \cup X'') \leq \max\{F(X'), F(X'')\} \qquad (5)$$

for arbitrary $X', X'' \subseteq U$ (one can interpret this and all succeeding versions of “quasiconvex” property as varieties of abstract quasiconvexity in accordance with [1]).

Theorem 3. *The function* $F(X)$ *is an ordered additive function if and only, if it is a monotone and* $\bigcup$*-quasiconvex.*

The proposition of the theorem is clear because the equality in the quasi-additive condition (2) is equivalent to the conjunction of two inequalities: a sub-additive condition (5) (i.e., $\bigcup$-quasiconvexity) and the super-additive condition:

$$F(X' \cup X'') \geq \max\{F(X'), F(X'')\} \qquad (6)$$

for arbitrary $X', X'' \subseteq U$ – what is equivalent to monotonicity (4).

We call $F(X)$ *hyper-additive*, if it is composed in the form of

$$F(X) = \max_{S \subseteq X} G(S) \tag{7}$$

for each $X \subseteq U$, where $G(S)$ – certain ordered function ("generator") on 2^U.

Lemma 1. *Hyper-additive* $F(X)$ *property is equivalent to its monotonicity*.

Theorem 4. *In consideration of* $F(X)$ *to be an ordered super-additive (sub-additive), it is necessary and sufficient for* F *to be composed in the form of*

$$F(X) = \max_{x \in X} \max_{S \subseteq X} g(x,S) \tag{8}$$

or, correspondingly

$$F(X) = \max_{x \in X} \min_{S \subseteq X} g(x,S) \tag{9}$$

for certain function g *on* $U \times 2^U$.

Notice 1. In semi-additive representations (8) (or (7)) for monotone, and (9), for $\bigcup$-quasiconvex function F, we might set up:

$g(x,S) \equiv G(S) \equiv F(S)$ – in (7) and (8), $g(x,S) \equiv \min_{Z: x \in Z \subseteq S} F(Z)$ – in (9).

Let call F $\bigcap$-quasiconvex, if

$$F(X' \cap X'') \leq \max\{F(X'), F(X'')\}. \tag{10}$$

Notice 2. Function $F(X)$ $\bigcap$-quasiconvexity is equivalent to function $F^c(X) \equiv F(U \setminus X)$ $\bigcup$-quasiconvexity.

Theorem 5. *For the function* $F(X)$ *to be* $\bigcap$*-quasiconvex it is necessary and sufficient that it can be represented in the form of*

$$F(X) = \max_{S \subseteq X} \max_{z \in U \setminus X} g(S,z) \tag{11}$$

with certain function g *on* $2^U \times U$.

Corollary (from Theorem 5 taking into account notice 2): $\bigcup$-quasiconvexity of function F is equivalent to its representation capability in the form

$$F(X) = \max_{x \in X} \max_{Z \subseteq U \setminus X} g(x,Z) \tag{12}$$

for certain function g on $U \times 2^U$ (concurrently with the representation (9)).

Let call F a $\bigcup,\bigcap$-quasiconvex, if it is at the same time $\bigcup$- and $\bigcap$-quasiconvex.

Theorem 6. For the function $F(X)$ to be $\bigcup,\bigcap$-quasiconvex it is necessary and sufficient that it can be represented in the form of

$$F(X) = \max_{x \in X} \max_{y \in U \setminus X} g(x,y) \tag{13}$$

with certain "flow" function $g(x,y)$ on $U \times U$.

Notice 3. In the representations (11), (12) and (13) it might be established, accordingly, that

$$g(S,z) = \min_{T: z \in T \subseteq U \setminus S} F(U \setminus T) = \min_{Q: z \notin Q, Q \supseteq S} F(Q), \tag{14}$$

$$g(x,Z) = \min_{T: x \in T \subseteq U \setminus Z} F(T) = \min_{Q: x \notin Q, Q \supseteq Z} F(U \setminus Q) \tag{15}$$

and $$g(x,y) = \min_{T: x \in T, y \notin T} F(T). \tag{16}$$

Lemma 2. *Every monotone (and only its like) functions F may be represented in the form of following decomposition along the system $\{g_v(x)\}_{v \in N}$ of quasi-additive functions on U:*

$$F(X) = \min_{v \in N} \max_{x \in X} g_v(x). \tag{17}$$

By virtue of the decomposition (17), as a corollary from theorems 4, 5, we obtain further equivalent functions representations.

Theorem 7. Following decompositions are valid:

1) For monotone (ordered super-additive) function F:

$$F(X) = \min_{v \in N} \max_{x \in X} \max_{y \in X} g_v(x,y); \tag{18}$$

2) For $\bigcup$-quasiconvex (ordered super-additive) function:

$$F(X) = \max_{v \in N} \max_{x \in X} \min_{y \in X} g_v(x,y) \tag{19}$$

or

$$F(X) = \max_{y \in U \setminus X} \min_{\mu \in M} \max_{x \in X} h_\mu(x,y); \tag{20}$$

3) For $\bigcap$-quasiconvex function:

$$F(X) = \max_{x \in X} \min_{v \in N} \max_{y \in U \setminus X} g_v(x,y) \tag{21}$$

– by means of certain collections of functions g_v (h_μ).

Notice 4. The framework to set an ordered function $F : 2^U \to L$ is adequate to setting a family $\{\mathrm{X}_\alpha\}_{\alpha \in L}$ of its level sets $\mathrm{X}_\alpha = \{X \subseteq U : F(X) \le \alpha\}$. One might examine:

1) that monotonicity of F is equivalent to every X_α set-theoretic shrinking result to be a member in its member-sets:

$$X \in \mathrm{X}_\alpha,\ S \subseteq X \Rightarrow S \in \mathrm{X}_\alpha; \tag{22}$$

2) $\bigcup$-quasiconvexity (either $\bigcap$-quasiconvexity) is equivalent for every X_α set-theoretic union (consecutively for an intersection) result to be a member in its member-sets:

$$\begin{gathered} X', X'' \in \mathrm{X}_\alpha \Rightarrow X' \cup X'' \in \mathrm{X}_\alpha \\ \text{(accordingly,} \Rightarrow X' \cap X'' \in \mathrm{X}_\alpha, \end{gathered} \tag{23}$$

i.e., equivalent for X_α to be a semilattice), therefore $\bigcup,\bigcap$-quasiconvexity of F is also equivalent for all X_α to be a lattice;

3) for the function F to be a quasi additive it is equivalent that each X_α constitutes a Boolean:

$$\mathrm{X}_\alpha = \{ X : X \subset X_\alpha \}, \text{ where } X_\alpha \subseteq U. \tag{24}$$

In particular case of ordered functions $F : 2^U \to L$ for $L = (0,1)$, i.e., for the logical set functions, the framework for every such function is adequate to the set X_0 "zeros" settlement, and herein the structure of such sets establishment emerge from F properties.

As an example of the above described ordered functions properties realization, we point to the "monotonic system" dataset model, which in the primitives of current work, according to J. Mullat [2] description, is a scalar function $F(X)$ of sets X defined on all subsets X within finite set of objects U. Function F definition is given by:

$$F(X) = \min_{x \in X} \pi(x, X), \tag{25}$$

where $\pi(x, X)$ – certain functions used in classification problems, whereas value $\pi(x, X)$ measures the object x "connectivity" within the set X; an effective algorithm for such category of F functions extreme points extraction may be found in [2]. Commencing the lemma 1 and the theorem 4 we conclude that an explicit monotonic system determines the function $F(X)$ in the sense of (25) if, and only if, the function $F(X)$ is an ordered quasi-sub-additive, in other words, if $F(X)$ is $\bigcup$-quasiconvex.

References

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