# Bargaining Solution on Boolean Tables

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**Abstract.** This article reports not only a theoretical solution to the bargaining problem as used by game theoreticians, but also provides pertinent calculation. An algorithm that can produce the result within a reasonable time frame is proposed, which can be performed computationally. The aim is to increase the current understanding of one nontrivial case of Boolean Tables.

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**Key words:** coalition; game; bargaining; algorithm; monotonic system [*]

> *"Rawls' second principle of justice: The welfare of the worst-off individual is to be maximized before all others, and the only way inequalities can be justified is if they improve the welfare of this worst-off individual or group. By simple extension, given that the worst-off is in his best position, the welfare of the second worst-off will be maximized, and so on. The difference principle produces a lexicographical ordering of the welfare levels of individuals from the lowest to highest."* Cit. Public Choice III, Dennis C. Mueller, p.600

## 1. Introduction

Since the publication of "The bargaining problem" by John F. Nash, Jr. in 1950, the framework proposed within has been developed in different directions. For example, in their "Bargaining and Markets" monograph, Martin Osborn and Ariel Rubinstein (1990) extended the "axiomatic" concept initially developed by Nash to incorporate a "strategic" bargaining process pertinent to everyday life. The authors posited that the "time shortage" is the major factor encouraging agreement between bargainers. Various bargaining problem varieties emerged in the decades following

[*] Monotonic Systems idea, different from all known ideas with the same name, was initially introduced in **1971** in the article of Tallinn Technical University Proceedings, Очерки по Обработке Информфции и Функциональному Фнализу, Seria A, No. 313, pp. 37-44.

Nash's pioneering work, prompting game theoreticians to seek their solutions, most of which did not necessarily comply with all Nash axioms. Beyond any doubt, "Nonsymmetrical Solution" proposed by Kalai (1977); Hursanyi's (1967) "Bargaining under Incomplete Information"; "Experimental Bargaining", which was later proposed by Roth (1985); and the "Bargaining and Coalition" paper published by Hart (1985) are among some notable contributions to this field, confirming the fundamental importance of bargaining theory.

Bargaining and rational choice mechanisms are interrelated concepts and are treated as such in this work. In the context of general choice theory, the choice act can be formalized through internal and external descriptions, which requires use of binary relations and theoretical approach, respectively. Thus, both description modes apply to the same object, albeit from different perspectives. The Nash Bargaining Problem and its solution express exactly the same phenomenon. Given a list of axioms, such as "Pareto Efficiency" or "Independence of Irrelevant Alternatives", in terms of binary relations the rational actors must follow, the solution is reached through scalar optimization applied to the set of alternatives. Indeed, the scalar optimization is at the core of the Nash's axiomatic approach and is the reason for its success in performing the bargaining solution calculation. In this respect, the motive of this work is to present a "calculation" of bargaining solution on large Boolean Tables and some theoretical foundations offered by the method. Unfortunately, in following the Nash's scenario, numerous difficulties emerged.

Boolean Table representation transforms the real life "cacophonous" scenario into a relatively simple and understandable data format. However, allowing the scalar optimization not to be unique makes the picture more complex. Moreover, we are considering a purely atomic object that does not intuitively satisfy the "invariance under the change of scale of utilities" property typically assumed in the proofs. From the researcher's point of view, the issue stems from the incertitude pertaining to the most optimal choice of the scalar criteria. The Nash axiomatic approach, however, suggests that employing the product of utilities is the most appropriate, thus removing any uncertainty from further discussion. Nevertheless, in the context of the method presented here, it is posited that a reasonable solution might come into consideration, while game-analysts would be advised to include the method into a wider range of applicable game analysis tools.

In the next section, the main example of our bargaining game is introduced. In addition, in the appendix, we also illustrate a different bargaining between the coalition and its moderator applied to Boolean Tables using some conventional characteristic functions. It is worth noting that certain items in the main example, such as signals or misrepresentations, are not the primary topic of our discussion. These items must rather be understood as an illustration of the bargaining process complexity. In Section 3, we attempt to explain our intentions in a more rigorous manner. Here, we formulate our "Bargaining Problem on Boolean Tables" in pure strategies, thus providing the foundation for Section 4, where we exploit our pure Pareto frontier in terms of so-called Monotonic Systems chain-nested alternatives—the Frontier Theorem. In order to implement the Nash theorem for nonsymmetrical solution (Kalai, 1977), in Section 5, we introduce what we deem to be an acceptable, albeit complex, algorithm in general form. Even though lottery is not permitted in the treatment of Boolean Tables subsets representing pure strategies, as this approach does not necessary produce the typical convex collection of feasible alternatives, we claim that the algorithm will yield an acceptable solution. Finally, Section 6 presents an elementary attempt to formulate a regular approach of coalition formation under the coalition formation supervisor–the moderator structure. This attempt depicted in Figure 2, explaining the notation nomenclature of chain-nested alternatives adopted in our Monotonic Systems theory, discussed in Section 4. Section 7 summarizes the entire analysis, while also providing an independent heuristic interpretation, before concluding the study in Section 8.

## 2. Example

Manager of the "Well-Being" company is determined to encourage employees to partake in health-promoting activities. The manger hopes to reduce company losses arising from disability compensations. To identify the employees' preferences, the manager has initiated a survey. According to the survey responses, five health activities offered to the employees generated varying degrees of interest, as shown in Table 1.

Table 1 Employee preferences pertaining to the company-sponsored health-promoting initiatives

| *Health activities* | *No Smoking* | *Swimming Pool* | *Bike Exercises* | *Moderate Alcohol* | *Fattening Diet* | *Total* |
|---|---|---|---|---|---|---|
| *Em. 1* | | x | x | | | 2 |
| *Em. 2* | x | x | | x | x | 4 |
| *Em. 3* | | x | x | x | | 3 |
| *Em. 4* | x | x | | x | x | 4 |
| *Em.. 5* | | | x | x | | 2 |
| *Em. 6* | x | x | x | x | x | 5 |
| *Em. 7* | | x | x | | | 2 |
| *Total* | 3 | 6 | 5 | 5 | 3 | 22 |

The manager would like to treat the responses the employees have provided as an indication that they are willing to partake in the activities they selected. However, aware of unreliable human nature, he is not confident that they will keep their promises. Therefore, the manager decides to award all employees that do participate in the health activities that will be organized in "Health Club". The manager has found a sponsor that has issued 12 Bank Notes in lieu of the project expenses. However, upon closer consideration of the awards policy, the manager realized that many obstacles must be overcome in order to implement it in practice.

Fist, organizing activities that only a few employees would partake in is neither practical nor cost-effective. Thus, it is necessary to stipulate a minimum number of employees that must subscribe to each health activity. On the other hand, it is desirable to promote all activities, encouraging the employees to attend them in greater numbers. For this initiative to be effective, instructions (as a rule full of twists and turns) regarding the awards regulations should be fair and concise. Usually, in such situations, someone (a moderator) must be in charge of the club formation and award allocation. However, as the manager is responsible for financing health activities, he/she should retain control of all processes. Thus, the manager proposes to write down the **First Club Regulation**: *The manager awards* 1 *Bank Note to an employee participating in at least* $k$ *different activities* (where $k$ is determined by the manager).

Determining the most optimal value of the parameter $k$ is not a straightforward task, as it is not strictly driven by employees' preferences regarding specific activities to participate in. In fact, this task is in the moderator's jurisdiction, while also being dependent on the employees' decisions, as they act as the club members. The goal is to prohibit some club members to "spring over" health activities preferred by other members of the club by worsening, in the manager's view, the situation, thus requiring too many different activities to be organized. This issue can be avoided by the inclusion of the **Second Club Regulation**: *If a certain employee in favor of receiving awards participated in fewer than* $k$ *activities, no one will be awarded.* By instituting this regulation, the manager aims to encourage the moderator to eliminate activities that would not have sufficient number of participants. Thus, the **Third Club Regulation**: *moderator's award basket will be equal to the lowest number of participants per activity in the list of activities among all actually participating club members*. Indeed, to earn more awards, the moderator might decide to organize a new club by excluding an activity with the lowest number of participants from the list of activities some of the members chose to attend as a part of the already organized club. This would effectively result in the lowest number of participants in the new and shorter list being higher than that in the previous list. It should also be noted that the award regulation does not address the situation in which a club member declines an activity, allowing an individual outside the club to participate instead. In such a case, the club "activities list" may become shorter than that presented in Table 1, and would determine the size of the moderator's award.

This scenario also provides the potential for the club members' preferences to be misrepresented to the company manager. Let us assume that the manager makes a decision $k = 1$, which has been, for whatever reason, made accessible to the moderator. Knowing that $k = 1$, the moderator actions can be easily predicted in accordance with the third club regulation. Indeed, using the employees' survey responses, the moderator can identify the most "popular" health activity, as well as the individuals that intend to participate in this activity. From the aforementioned regulations, it is evident that the moderator would receive the maximum award if he manages to persuade other employees to participate in that particular activity only. Rational members would certainly agree to that proposal because, whether or not they take part in any other activity, their award is still guaranteed.[1] The same logic obviously applies for $k > 1$ as well.

Thus, the essence of establishing fair rules pertains to determining the moderator's award. If the moderator is not offered any awards, the grand coalition formation is guaranteed, as all employees will become club members. This is the case, as participating in at least one activity would ensure that an employee receives an award. However, due to the moderator actions, such grand coalition formation is not always feasible.

As previously noted, the moderator might receive a minor award if a "curious" employee decides to take part in an "unpopular activity". Indeed, the third club regulation stipulates that the number of participants in the most "unpopular activity" governs the moderator award size. Being

[1] We will disclose more complex misrepresentation opportunity later.

aware of the potential manipulation of the regulations, and being a rational actor, the company manager will thus strive to keep the decision $k$ a secret. It is also reasonable to believe that all parties involved—the club members, the moderator and the manager—will have their own preferences regarding the value of $k$. Therefore, an explanation based on the salon game principles is applicable to this scenario. Using this analogy, let us assume that the manager has chosen a card $k$ and has hidden it from the remaining players. Let us also assume that the moderator and the club members have reached an agreement on their own card choice in line with the three aforementioned club regulations. The game terminates and awards are paid out only if their chosen card is higher than that selected by the manager. Otherwise, no awards will be paid out, despite taking into consideration the club formation.

However, not all factors affecting the outcome have been considered above. Indeed, the positive effect, $f_k$, which the manager hopes to achieve, depends on the decision $k$. We have to expect a single $\cap$-peakedness of the effect function for some reason. As a result, this function separates the region of $k$ values into what we call prohibitive and normal range. In the prohibitive range, which includes the low k values, the effect has not yet reached its maximum value. On the other hand, when k value is high (i.e. in the normal range), the $f_k$ limit is exceeded. Therefore, in the prohibitive range, the manager and the moderator interests compete with each other, making it reasonable to assume that the manager would keep his/her decision a secret. However, in the normal range, they might cooperate, as neither benefits from very high $k$ values, given that both can lose their payoffs. Consequently, using the previous card game analogy, in the normal range, it is not in the manager's best interests to hide the $k$ card.

Given the arguments presented above, the game scenario can be illustrated more precisely. Using the data presented in Table 1, and assuming that an award will be granted at $k = 1,2$, the manager may count upon all seven employees to become the club members. If all employees participate in all activities, each would receive a Bank Note, and the moderator's basket size would be equal to 3. However, it would be beneficial for the moderator to entice to the club members to decline participation in *"No Smoking"* and *"Fattening Diet"* activities, as this would increase his/her own award to 5. As all club members will still preserve their awards, they have no reason not to support the moderator's suggestion, as shown in Table 2.

Table 2

| *Health activities* | *Swimming Pool* | *Bike Exercises* | *Moderate Alcohol* | *Total* |
|---|---|---|---|---|
| *Em. 1* | x | x | | 2 |
| *Em.. 2* | x | | x | 2 |
| *Em.. 3* | x | x | x | 3 |
| *Em.. 4* | x | | x | 2 |
| *Em. 5* | | x | x | 2 |
| *Em.. 6* | x | x | x | 3 |
| *Em.. 7* | x | x | | 2 |
| *Total* | 6 | 5 | 5 | 16 |

Table 3

| *Swimming Pool* | *Total* |
|---|---|
| x | 1 |
| x | 1 |
| x | 1 |
| x | 1 |
| | 0 |
| x | 1 |
| x | 1 |
| 6 | 6 |

In this scenario, the sponsor would have to issue 12 Bank Notes, which can be treated as expenses associated with organizing the club. The sponsor may also conclude that $k=1$ is undesirable based on the previous observation that the moderator can deliberately misrepresent the members' preferences for personal gain.[2] The sponsor is aware that the moderator may offer one Bank Note to an employee that agrees to propose $k=1$. Knowing that $k=1$, the moderator may suggest to the club members to subscribe to the *"Swimming Pool"* activity only. However, in the sponsor's opinion, the moderator must compensate Employee no. 5 for the losses incurred by offering him/her one Bank Note. Otherwise, Employee no. 5, by participating in other activities distinct from *"Swimming Pool"* has the right to receive an award and may report the moderator's fraud to the board. In this case, following the regulations in force (see Table 3), moderator's award will be equal to 4 (1 would be deducted for the signal and 1

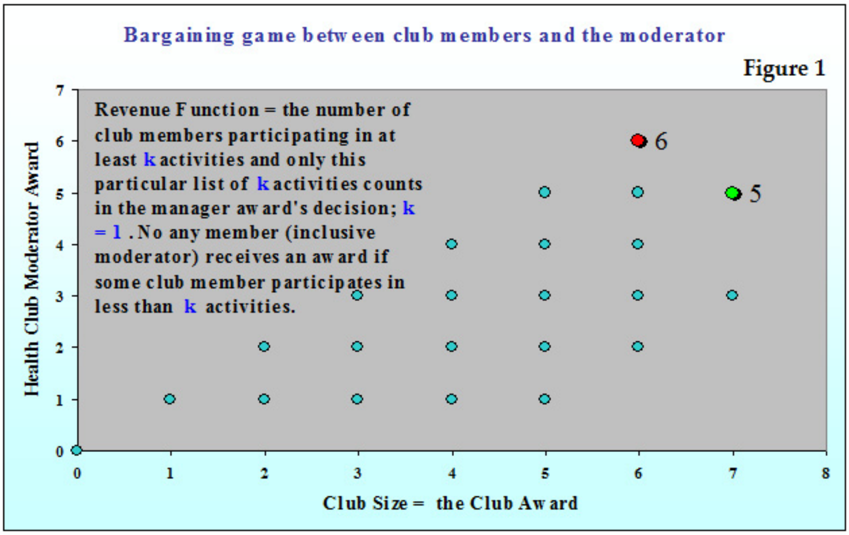
[2] The more complex case of misrepresentation follows, as promised.

for the compensation). However, this would still exceed the value indicated in Table 1. Thus, in order to decrease sponsor expenses or avoid misrepresentations, the company board may follow the sponsor's advice and propose $k \geq 3$ .

It could be argued that $k \geq 3$ results in decreased participation in health activities because Employees no. 1, 5 and 7 will be excluded from the club and will immediately cease to partake in any of their initially chosen activities. However, based on Table 4, it can also be noted that, in such an event, the remaining employees (i.e. 2,3,4 and 6) will still participate in heath activities and will still be awarded.

Table 4

| *Health activities* | *No Smoking* | *Swimming Pool* | *Bike Exercises* | *Moderate Alcohol* | *Fattening Diet* | *Total* |
|---|---|---|---|---|---|---|
| *Em. 2* | x | x | | x | x | 4 |
| *Em. 3* | | x | x | x | | 3 |
| *Em. 4* | x | x | | x | x | 4 |
| *Em. 6* | x | x | x | x | x | 5 |
| *Total* | 3 | 4 | 2 | 4 | 3 | 16 |



Now, the moderator's award basket is equal to $2$, since only Employees no. $3$ and $6$ would take part in *"Bike Exercises"*. Consequently, the sponsor expenses decrease from $10$ to $6$. In this case, the manager may decide to allow the moderator to retain his/her award of $3$ by eliminating *"Bike Exercises"* from the activity list, as organizing it for two participants only is not justified, as shown in Table 5. Note that Employee no. 3, due to this decision, must be excluded from the club list, in line with the second club regulation, c.f. the suggestion above to eliminate *"No Smoking"* and *"Fattening Diet"* activities.

Table 5

| *Health activities* | *No Smoking* | *Swimming Pool* | *Moderate Alcohol* | *Fattening Diet* | *Total* |
|---|---|---|---|---|---|
| *Em. no. 2* | x | x | x | x | 4 |
| *Em. no. 4* | x | x | x | x | 4 |
| *Em. no. 6* | x | x | x | x | 4 |
| *Total* | 3 | 3 | 3 | 3 | 12 |

This decision does not seem reasonable, given that the aim of the initiative was to motivate the employees to exercise and improve their health. Thus, let us assume that $k = 5$ was the board proposal. This result would only concern Employee no. 6 being willing to participate in the health activities offered, see Table 6.

Table 6

| *Health activities* | *No Smoking* | *Swimming Pool* | *Bike Exercises* | *Moderate Alcohol* | *Fattening Diet* | *Total* |
|---|---|---|---|---|---|---|
| *Em. 6* | x | x | x | x | x | 5 |
| *Total* | 1 | 1 | 1 | 1 | 1 | 5 |

The moderator may decide not to organize the club, as this would result in an award equal to only one Bank Note. Similarly, the manager is not incentivized to promote all five activities if only one employee would take part in each one. As a result, at the board meeting, the manager would vote against the proposal $k = 5$. In sum, the manager's dilemma pertains to the alternative $k$ choice based on the information given in Table 7.

Table 7.

| | *Club members* | *Club moderator* | *Club members compensation* | *Sig-nal* | *Bank Notes used* | *Bank Notes left* |
|---|---|---|---|---|---|---|
| T. 1, $k=2$ | 7 | 3 | 0 | 0 | 10 | 2 |
| T. 2, $k=2$ | 7 | 5 | 0 | 0 | 12 | 0 |
| T. 3, $k=1$ | 6 | 4 | 1 | 1 | 12 | 0 |
| T. 4, $k=4$ | 3 | 1 | 0 | 0 | 4 | 8 |
| T. 5, $k=4$ | 3 | 3 | 0 | 0 | 6 | 6 |
| T. 6, $k=5$ | 1 | 1 | 0 | 0 | 2 | 10 |

To clarify the situation presented in tabular form, it would be helpful to visualize the manager's dilemma using the bargaining game analogy, where 12 Bank Notes are shared between the moderator and the club members.

The decision on the most optimal $k$ value taken at the board meeting will be revealed later, using rigorous nomenclature, as only a closing topic is necessary to interrupt our pleasant story for a moment.[3]

Let us assume that three actors are engaged in the bargaining game: $N$ employees, one moderator in charge of club formation, and the manager. Certain employees from $N=\{1,...,i,...n\}$ – the potential members of the club $x$, $x\in 2^N$, have expressed their willingness to participate in certain activities $y$, $y\in 2^M$, $M=\{1,...,j,...,m\}$. Let a Boolean Table $W=\|a_{ij}\|_n^m$ reflect the survey results pertaining to employees' preferences, whereby $a_{ij}=1$ if employee $i$ has promised to participate in activity $j$, and $a_{ij}=0$ otherwise. In addition, $2^M$ denotes of allegedly subsidized activities, whereby $y\in 2^M$ have been examined.

3 Those unwilling to continue with the discussions on bargaining presented in the subsequent sections should nonetheless pay attention to this closing remark.

We can calculate the moderator payoff $F_k(H)$ using a sub-table $H$ formed by crossing entries of the rows $x$ and columns $y$ in the original table $W$ by further selection of a column with the least number $F_k(H)$ from the list $y$. The number of 1-entries in each column belonging to $y$ determines the payoff $F_k(H)$. Characteristic functions family $\nu^k(x,y) \equiv \nu^k(H)$, $k \in \{1,...,k,...,k_{max}\}$, on $N$ are known for the coalition games; in particular, for every pair $L \subset G$, $L,G \in 2^N \times 2^M$, we suppose that $\nu^k(L) \leq \nu^k(G)$. Further assuming that the manager payoff function $f_k(H)$ has a single $\cap$-peakedness, in line with the decisions $\langle 1,...,k,...,k_{max} \rangle$, $f_k(H)$ reflects some kind of positive effect on the company deeds. In this case, sponsor expenses will be equal to $\nu^k(H) + f_k(H)$.

Finally, it is appropriate to share some ideas regarding a reasonable solution of our game. The situation is similar to the Nash Bargaining Problem first introduced in 1950, where two partners—the club members and the moderator—are striving to reach a fair agreement. It is possible to find the Bargaining Solution $S_k \in \{H\} = 2^N \times 2^M$ for each particular decision $k$, see next sections. However, the choice of the number $k$ is not straightforward, as previously discussed. For example, $k = 4,5$ may be useful based on some *ex ante* reasoning, whereas maximum payoffs are guaranteed for the partners when $k = 1$. As that decision is irrational, because only one activity will be organized and, even though it will attract the maximum number of participants, it would fail to yield a positive effect $f(S_k)$ on the health deeds in general. The choice of higher $k$ was previously shown to be counterproductive (too many activities will be offered, but would have only a few participants), yet the sponsor would benefit from issuing fewer awards. For example, for $k = k_{max}$, an employee with the largest number of preferred $k_{max}$ activities might become the only member of the club. This is akin to the median voter scheme, discussed by Barbera et al. (1993). However, a further consultation in this “white field” is necessary.

## 3. BARGAINING GAME APPLIED TO BOOLEAN TABLES

Suppose that employees who intend to participate in company activities have been interviewed in order to reveal their preferences. The resulting data can be arranged in $n \times m$ table $W = \|\alpha_{ij}\|$, where the entry $\alpha_{ij} = 1$ indicates that an employee $i$ has promised to participate in activity $j$, otherwise $\alpha_{ij} = 0$. In this respect, the primary table $W$ is a collection of Boolean columns, each of which comprises of Boolean elements related to one specific activity. In the context of the bargaining game, we can discuss an interaction between the health club and the moderator. The club choice $x$ is a subset of rows $\langle 1,...,i,...,n \rangle$ denoting the newly recruited club members, whereby a subset $y$ of columns $\langle 1,..., j,...m \rangle$ is the moderator's choice—the list of available activities. The result of the interaction between the club and the moderator can thus represent a sub-table $H$ or a block, denoting the players' joint anticipation $(x, y)$. The players are designated as Player no. 1 – the club, and Player no. 2 – the moderator, and both are driven by the desire to receive the awards. Let us assume that all employees have approved our three award regulations.[4] While both players are interested in company activities, their objectives are different. Player no. 1 might aim to motivate each club member to agree to partaking in a greater number of company-sponsored activities. Player No. 2, the moderator, might desire to subscribe maximum number of participants in each activity arranged by the company. Let the utility pair $(v(x), F(y))$ denote the players' payoff, whereby both players will bargain upon all possible anticipated outcomes $(v, F)$.

Our intention in developing a theoretical foundation for our story was to follow the Nash's (1950) axiomatic approach. Unfortunately, as previously observed, some fundamental difficulties arise when adopting similar approach. Below, we summarize each of these difficulties, and propose

[4] We recall the main regulation that none of the club members, inclusive the moderator, receive their awards if a certain club member participates in fewer than $k$ activities.

a suitable equivalent. When proceeding in this direction, we first formulate the Nash's axioms in their original nomenclature before reexamining their essence in our own nomenclature. This approach would allow us to provide the necessary proofs in the sections that follow.

As noted by Nash (1950), "... we may define a two-person anticipation as a combination of two one-person anticipation. ... A probability combination of two two-person anticipations is defined by making the corresponding combinations for their components" (p. 157). Readers are also advised to refer to Sen Axiom 8*1, p. 127, or sets of axioms, as well as Luce and Raiffa (1958), Owen (1968) and von Neumann and Morgenstern (1947), with the latter being particularly relevant for utility index interpretation. Rigorously speaking, the compactness and convexity of a feasible set $\boldsymbol{S}$ of utility pairs ensures that any continuous and strictly convex function on $\boldsymbol{S}$ reaches its maximum, while convexity guarantees the maximum point uniqueness.

Let us recall the other Nash axioms. The solution must comply with INV (invariance under the change of scale of utilities); IIA (independence of the irrelevant alternatives); and PAR (Pareto efficiency). Note that, following PAR, the players would object to an outcome $s$ when an outcome $s'$ that would make both of them better off exists. We expect that the players would act from a *strong individual rationality* principle SIR. An arbitrary set $\boldsymbol{S}$ of the utility pairs $s=(s_1,s_2)$ can be the outcome of the game. A disagreement arises at the point $d=(d_1,d_2)$ where both players obtain the lowest utility they can expect to realize – the *status quo* point. A *bargaining problem* is a pair $\langle \boldsymbol{S}, d\rangle$[5] and there exists $s \in \boldsymbol{S}$ such that $s_i > d_i$ for $i=1,2$ and $d \in \boldsymbol{S}$. A *bargaining solution* is a function $f(\boldsymbol{S},d)$ that assigns to every bargaining problem $\langle \boldsymbol{S}, d\rangle$ a unique element of $\boldsymbol{S}$. The bargaining solution $f$ satisfies SIR if $f(\boldsymbol{S},d) > 0$ for every bargaining problem $\langle \boldsymbol{S}, d\rangle$.

[5] We use the bold notifications $\boldsymbol{S}$ close to the originals. Notification $S$ is preserved for stable point, see later.

The advantage of our approach, which guarantees the same properties, lies in the following. We define a feasible set $\boldsymbol{S}$ of anticipations, or in more convenient nomenclature, a feasible set $\boldsymbol{S}$ of alternatives as a collection of table $\mathrm{W}$ blocks: $\boldsymbol{S} \subseteq 2^{\mathrm{W}}$. Akin to the disagreement event in the Nash scheme, we define an empty block $\varnothing$ as a *status quo* option in any set of alternatives $\boldsymbol{S}$, which we call the refusal of choice. Given any two alternatives $\mathrm{H}$ and $\mathrm{H}'$ in $\boldsymbol{S}$, an alternative $\mathrm{H} \cup \mathrm{H}'$ belongs to $\boldsymbol{S}$. In other words, in our case, the set $\boldsymbol{S}$ of feasible alternatives always forms an upper semi-lattice. Moreover, if an alternative $\mathrm{H} \in \boldsymbol{S}$, it follows that all of its subsets $2^{\mathrm{H}} \subseteq \boldsymbol{S}$. Although these arguments do necessitate further discussion, at this juncture, we will state that this is our equivalent to the convex property and will play the same role in proofs as it does in the Nash scheme.

The Nash theorem asserts that there is a unique bargaining solution $\mathrm{f}(\boldsymbol{S},\mathrm{d})$ for every bargaining problem $\langle \boldsymbol{S},\mathrm{d} \rangle$, which maximizes the product of the players' gains in the set $\boldsymbol{S}$ of utility pairs $(\mathrm{s}_1,\mathrm{s}_2) \in \boldsymbol{S}$ over the disagreement outcome $\mathrm{d} = (\mathrm{d}_1,\mathrm{d}_2)$. This is a so-called symmetric bargaining solution, which satisfies INV, IIA, PAR, and SYM – players symmetric identify, if and only if

$$\mathrm{f}(\boldsymbol{S},\mathrm{d}) = \arg\max_{(\mathrm{d}_1,\mathrm{d}_2) \leq (\mathrm{s}_1,\mathrm{s}_2)} (\mathrm{s}_1 - \mathrm{d}_1) \cdot (\mathrm{s}_2 - \mathrm{d}_2). \tag{1}$$

It is difficult to make an *ad hoc* assertion regarding properties that can guarantee the uniqueness of similar solution on Boolean Tables. Nevertheless, in the next section, we claim that our bargaining problem on $\boldsymbol{S} \subseteq 2^{\mathrm{W}}$ has the same symmetric or nonsymmetrical shape:

$$\mathrm{f}(\boldsymbol{S},\varnothing) \equiv \mathrm{f}(\boldsymbol{S}) = \arg\max_{\mathrm{H} \in \boldsymbol{S}} \mathrm{v}(\mathrm{H})^{\theta} \mathrm{F}(\mathrm{H})^{1-\theta} \tag{2}$$

for some $0 \leq \theta \leq 1$ provided that Nash axioms hold.

## 4. THEORETICAL ASPECTS OF THE BOOLEAN GAME

Henceforth, the table $W = \|\alpha_{ij}\|$ will denote the Boolean table discussed in the preceding section, representing employees' promises to attend company activities. It is beneficial to examine $H$ rows $x$, symbolizing the arrival of new members to the club, committed to participating in at least $k$ activities. Activities form, what we call here, a column's activity list $y$, $k = 2,3,\ldots$, where $k$ represents the award decision. For each activity in the activity list $y$, at least $F(H)$ of club members intend to fulfill their promises. For example, let us consider the number of rows in $H$ pertaining to the gain $v(H)$ of Player no. 1 (the club members), while the gain of Player no. 2 (the moderator's award) is represented by $F(H)$.

Let us look at the bargaining problem in conjunction with players' preferences. The anticipations of the coming club members $i \in x$ towards the activity list $y$ can easily be "raised" by $r_i = \sum_{j \in y} \alpha_{ij}$ if $r_i \geq k$, and $r_i = 0$ if $\sum_{j \in y} \alpha_{ij} < k$, $i \in x$, $j \in y$. Similarly, the moderator's anticipation towards the same activity list $y$ can be "accumulated" by means of table $H$ as $c_j = \sum_{i \in x} \alpha_{ij}$, $j \in y$.

We now consider this scenario in more rigorous mathematical form. Below, we use the notation $H \subseteq W$. The notation $H$ contained in $W$ will be understood in an ordinary set-theoretical nomenclature, where the Boolean Table $W$ is a set of its Boolean 1-elements. All 0-elements will be dismissed from the consideration. Thus, $H$ as a binary relation is also a subset of $W$. Henceforth, when referring to an element, we assume that it is a Boolean 1-element.

For an element $\alpha \equiv \alpha_{ij} \in W$ in the row $i$ and column $j$, we use the similarity index $\pi_{ij} = c_j$, counting only on the Boolean elements belonging to $H$, $i \in x$ and $j \in y$. As the value of $\pi_{ij} = c_j$ depends on each subset $H \subseteq W$, we may write $\pi_{ij} \equiv \pi \equiv \pi(\alpha, H)$, where the set $H$ represents the $\pi$-function parameter. It is evident that our similarity indices $\pi_{ij}$ may only increase with the "expansion" and decrease with the "shrinking" of the parameter $H$. This yields the following fundamental definitions:

**Definition 1**. Basic monotone property. *Monotonic System will be understood as a family* $\{\pi(\alpha, H) : H \in 2^W\}$ *of* $\pi$*-functions, such that the set* $H$ *is a parameter with the following monotone property: for two particular values* $L, G \in 2^W$, $L \subset G$ *of the parameter* $H$, *the inequality* $\pi(\alpha, L) \le \pi(\alpha, G)$ *holds for all elements* $\alpha \in W$. In ordinary nomenclature, the $\pi$*-function* with the definition area $W \times 2^W$ is monotone on $W$ with regard to the second parameter on $2^W$.

**Definition 2**. Let $V(H)$ for a non-empty subset $H \subseteq W$ by means of a given arbitrary threshold $u$ be the subset $V(H) = \{\alpha \in W : \pi(\alpha, H) \ge u\}$. *The non-empty* $H$*-set indicated by* $S$ *is called a stable point with reference to the threshold* $u$ *if* $S = V(S)$ *and there exists an element* $\xi \in S$, *where* $\pi(\xi, S) = u$. See Mullat (1979, 1981) for a comparable concept. Stable point $S = V(S)$ has some important properties, which will be discussed later.

**Definition 3**. *By Monotonic System kernel we understand a stable point* $S^* = S_{max}$ *with the maximum possible threshold value* $u^* = u_{max}$.

Libkin et al. (1990), Genkin et al. (1993), Kempner et al. (1997), and Mirkin et al. (2002) have investigated similar properties of Monotonic Systems and their kernels. With regard to the current investigation, it is noteworthy to state that, given a Monotonic System in general form, without any reference to any kind of "interpretation mechanism", one can always consider a bargaining game between a coalition $H$ – Player no. 1, with characteristic function $v(H)$, and Player no. 2 with the payoff function $F(H) = \min_{\alpha \in H} \pi(\alpha, H)$. Following Nash theorem, a symmetrical solution has to be found in form (1). In addition, we will prove below that our solution has to be found in the symmetrical or nonsymmetrical form (2).

**Definition 4**. Let $d$ be the number of Boolean 1's in table $W$. *An ordered sequence* $\overline{\alpha} = \langle \alpha_0, \alpha_1, ..., \alpha_{d-1} \rangle$ *of distinct elements in the table* $W$ *is called a defining sequence if there exists a sequence of sets* $W = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *such that:*

A. Let the set $H_k = \{\alpha_k, \alpha_{k+1}, ..., \alpha_{d-1}\}$. *The value* $\pi(\alpha_k, H_k)$ *of an arbitrary element* $\alpha_k \in \Gamma_j$, *but* $\alpha_k \notin \Gamma_{j+1}$ *is strictly less than* $F(\Gamma_{j+1})$, $j = 0,1,..., p-1$.
B. *There does not exist in the set* $\Gamma_p$ *a proper subset* $L$ *that satisfies the strict inequality* $F(\Gamma_p) < F(L)$.

**Definition 5.** *A defining sequence is complete, if for any two sets* $\Gamma_j$ *and* $\Gamma_{j+1}$ *it is impossible to find* $\Gamma'$ *such that* $\Gamma_j \supset \Gamma' \supset \Gamma_{j+1}$ *while* $F(\Gamma_j) < F(\Gamma') < F(\Gamma_{j+1})$, $j = 0,1,..., p-1$.

It has been established that, in an arbitrary Monotonic System, one can always find a complete defining sequence (see Mullat, 1971, 1976). Moreover, each set $\Gamma_j$ is the largest stable set with reference to the threshold $F(\Gamma_j)$. This allows us to formulate our Frontier Theorem.

**Frontier Theorem.** *Given a bargaining game on Boolean Tables with an arbitrary set* $\boldsymbol{S}$ *of feasible alternatives* $H \in \boldsymbol{S}$, *the anticipations points* $(v(\Gamma_j), F(\Gamma_j))$, $j = 0,1,..., p$, *of a complete defining sequence* $\overline{\alpha}$ *arrange a Pareto frontier in* $\Re^2$.

*Proof*. Let $W^S \in \boldsymbol{S}$ be the largest set in $\boldsymbol{S}$ containing all other sets $H \in \boldsymbol{S}: H \subseteq W^S$. Let a complete defining sequence $\overline{\alpha}$[6] exist for $W^S$. Let the set $H^c$ be the set containing all such sets $V(H)$, where $V(H) = \{\alpha \in W : \pi(\alpha, H) \geq F(H)\}$. Note that $H \subseteq V(H^c)$ and $F(H^c) \geq F(H)$. Now, for accuracy, we must distinguish three situations: (a) in the sequence $\overline{\alpha}$ one can find an index j such that $F(\Gamma_j) \leq F(H^c) < F(\Gamma_{j+1})$ $j = 0,1,\dots,p-1$; (b) $F(H^c) < F(W) = F(\Gamma_0)$; and (c) $F(H) > F(\Gamma_p)$. The case (c) is impossible because, on the set $\Gamma_p$, the function $F(H)$ reaches its global maximum. In case of (b), the anticipation $(v(\Gamma_0), F(\Gamma_0))$, $\Gamma_0 = W$, is more beneficial than $(v(H), F(H))$, which concludes the proof. In case of (a), let $F(\Gamma_j) < F(H^c)$, otherwise the equality $F(\Gamma_j) = F(H^c)$ is the statement of the theorem (when reading the sentence after the next, the index $j+1$ should be replaced by $j$). However, in this case, the set $H^c$ must coincide with $\Gamma_{j+1}$, $j = 0,1,\dots,p-1$, otherwise the defining sequence $\overline{\alpha}$ is incomplete. Indeed, looking at the first element $\alpha_k \in H^c$ in the sequence $\overline{\alpha}$, it can be ascertained that, if $\Gamma_{j+1} = H^c$ does not hold, the set $H_k = H^c$ because it is the largest stable set up to the threshold $F(H^c)$. Hence, the set $H_k$ represents an additional $\Gamma$-set in the sequence $\overline{\alpha}$ with the property A of a complete defining sequence. The inequalities $F(\Gamma_{j+1}) = F(H^c) \geq F(H)$ and $v(\Gamma_{j+1}) = v(H^c) \geq v(H)$, due to $\Gamma_{j+1} = H^c \supseteq H$ and the basic monotonic property, are true. Thus, the point $(v(\Gamma_{j+1}), F(\Gamma_{j+1}))$ is more advantageous than $(v(H), F(H))$. ■

[6] We are not going to use any new notifications to distinguish between Boolean Tables $W$ and $W^S$.

## 5. CALCULATION OF THE BARGAINING SOLUTION

To summarize, the discussion that follows is governed by the Nash bargaining scheme. Some reservations (see, for example, Luce and Raiffa, 6.6) hold as usual because our bargaining game on Boolean Tables is purely atomic, i.e. it does not permit lotteries (which are an important element of any bargaining scenario). Given this restriction, the uniqueness of the Nash solution cannot be immediately guaranteed. However, it is important to note that "...the Nash solution of $\langle \boldsymbol{S}, \mathrm{d} \rangle$ depends only on disagreement point $\mathrm{d}$ and the Pareto frontier of $\boldsymbol{S}$. The compactness and convexity of $\boldsymbol{S}$ are important only insofar as they ensure that the Pareto frontier of $\boldsymbol{S}$ is well defined and concave. Rather than starting with the set $\boldsymbol{S}$, we could have imposed our axioms on a problem defined by a non-increasing concave function (and disagreement point $\mathrm{d}$ …Osborn and Rubinstein, 1990, p. 24). In our case, $\left(\mathrm{v}(\Gamma_{\mathrm{j}}), \mathrm{F}(\Gamma_{\mathrm{j}})\right)$, $\mathrm{j} = 0,1,\ldots,\mathrm{p}$, represents the atomic Pareto frontier. Therefore, it is possible to provide the proof of non-symmetrical solution (see Kalai, 1977, p. 132), as well as perform the calculation with the product of utility gains in its asymmetrical form (2).[7] The problem of maximizing the product is primarily of technical nature. In the discussions that follow, we will introduce an algorithm for that purpose. We will first comment on the individual algorithm step in relation to the definitions.

The algorithm's last iteration, see below, through the step **T** detects the largest kernel $\overline{\mathrm{K}} = \mathrm{S}^{*}$[8] (Mullat, 1995). The original version (Mullat, 1971) of the algorithm aimed to detect the largest kernel and is akin to a greedy inverse serialization procedure (Edmonts, 1971). The original version of

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[7] There are many techniques that guarantee the uniqueness of the product of utility gains. We are not going to discuss this matter here, because this case is rather an exemption than a rule.

[8] It is possible that some smaller kernels exist as well.

the algorithm produces a complete defining sequence, which is imperative for finding the bargaining solution aligned with the Frontier Theorem. In the context of the current version, however, it fails to produce a complete defining sequence. Rather, it only detects some thresholds $u_j$, and some stable set $\Gamma_j = S_j$. The sequence $u_0, u_1, \ldots$ is monotonically increasing: $u_0 < u_1 < \ldots$ while the sequence $\Gamma_0, \Gamma_1, \ldots$ is monotonically shrinking: $\Gamma_0 \supset \Gamma_1 \supset \ldots$, whereby the set $\Gamma_0 = W$ is stable towards the threshold $u_0 = F(W) = \min_{(i,j)\in W} \pi_{ij}$. Hence, the original algorithm is always characterized by higher complexity. However, for finding the bargaining solution, we can still implement an algorithm of lower complexity, which would require modifying the indices $\pi_{ij} = c_j$.

Let us consider the problem of identifying the players' joint choice $H_{max}$ representing a block $\arg\max_{H\in\boldsymbol{S}} v(H)^{\theta} F(H)^{1-\theta}$ of the rows $x$ and columns $y$ in the original table $W$ satisfying the property $\sum_{j\in y} \alpha_{ij} \geq k$, $i \in x$.

Let an index $\pi_{ij} = r_i \cdot v^{\theta} \cdot c_j^{1-\theta}$[9]. The following algorithm solves the problem.

**Algorithm.**

**Step I.** Set the initial values.

**1i.** Assign the table parameter $H$ to be identical with $W$, $H \Leftarrow W$. Set the minimum and maximum bounds $a, b$ on the threshold $u$ for $\pi_{ij} \in H$ values.

**Step A.** Establish that the next step **B** produces a non-empty sub-table $H$. Remember the current status of table $H$ by creating a temporary table $H°$: $H° \Leftarrow H$.

**1a.** Test $u$ as $(a+b)/2$ using step **B**. If it succeeds, replace $a$ by $u$, otherwise replace $b$ by $u$ and $H$ by $H°$: $H \Leftarrow H°$ - "regret action".

**2a.** Go to **1a**.

[9] This index obeys the basic monotone property as well.

**Step B.** Test whether the minimum of $\pi_{ij} \in H$ over $i, j$ can be equal or greater than $u$.

**1b.** Delete all rows in $H$ where $r_i = 0$. This step **B** fails if all rows in $H$ must be deleted, in which case proceed to **2b**. The table $H$ is shrinking.

**2b.** Delete all elements in columns where $\pi_{ij} \le u$. This step **B** fails if all columns in $H$ must be deleted, in which case proceed to **3b**. The table $H$ is shrinking.

**3b.** Perform step **T** if no deletions were made in **1b** and **2b**; otherwise go to **1b**.

**Step T.** Test whether the global maximum is found. Table $H$ has halted its shrinking.

**1t.** Among numbers $\pi_{ij} \in H$, find the minimum $\min \leftarrow \pi_{ij}$ and then perform Step **B** with new value $u = \min$. If it succeeds, set $a = \min$ and return to Step **A**; otherwise, terminate the algorithm.

## 6. BOOLEAN GAME COOPERATIVE ASPECTS

A cooperative game is a pair $(N, v)$, where $N$ symbolizes a set of players and $v$ is the game characteristic function. Function $v$ is called a super modular if $v(L) + v(G) \le v(L \cup G) + v(L \cap G)$ whereas it is sub modular if the inequality sign $\le$ is replaced by $\ge$, $L, G \in 2^N$. Among others, see Cherenin et al. (1948) and Shapley (1971), where various properties of supermodular set functions are specified. In the appendix, we illustrate a game, which is neither supermodular nor submodular, but rather a mixture of the two, where single and pairwise players do not receive extra awards. On the other hand, it is obvious that all properties of supermodular functions $v$ remain unchanged for submodular $-v$ characteristic function or vice versa.

A marginal contribution into the coalition $H$ of a player $x$ (the player marginal utility) is given by $\pi(x; H) \equiv \frac{\partial H}{\partial x}$, where

$$\frac{\partial H}{\partial x} = v(H \cup x) - v(H)$$

if $x \notin H$, the player $x$ joins the coalition, and

$$\frac{\partial H}{\partial x} = v(H) - v(H \setminus x)$$

if $x \in H$, the player $x$ leaves the coalition, for every $H \in 2^W$. We denote in our nomenclature $H \cup x \equiv H + x$, and $H \setminus x \equiv H - x$, see later.

Suppose that the interest of player $x$ to join the coalition equals the player's marginal contribution $\frac{\partial H}{\partial x}$. A coalition game is convex (concave) if for any pair $L$ and $G$ of coalitions $L \subseteq G \subseteq W$ the inequality

$$\frac{\partial L}{\partial x} \leq \frac{\partial G}{\partial x} \left( \frac{\partial L}{\partial x} \geq \frac{\partial G}{\partial x} \right) \text{ holds for each player } x \in W.$$

**Theorem.** *For the coalition game to be convex (concave) it is necessary and sufficient for its characteristic function to be a supermodular (submodular) set function.*

Extrapolated from Nemhauser et al. (1978).[10]

Now, in view of the theorem, marginal utilities of players in the supermodular game motivate them in certain cases to form coalitions. In a modular game, where the characteristic function is both supermodular and submodular, marginal utilities are indifferent to collective rationality because entering a coalition would not allow anybody to win or lose their respective payments. In contrast, it can be shown that collective rationality is sometimes counterproductive in submodular games. Therefore, in supermodular games, formation of too many coalitions might be unavoidable, resulting in, for example, the grand coalition. In such cases, in

[10] Shapley (**1971**) recognized this condition as equivalent, whereby Nemhauser et al. (**1978**) proposed similar derivatives in their investigation of some optimization problems. Muchnik and Shvartser (**1987**) pointed to the link between submodular set functions and the Monotonic Systems, see Mullat (**1971**).

Shapley's (1971) words, this leads to a "snowballing" or "band-wagon" effect. On the other hand, submodular games are less cooperative. In order to counteract these "bad motives" of players in both supermodular and submodular games, we introduce below a second actor – the moderator. Hence, we consider a bargaining game between the coalition and the moderator.

Convex game induces an accompanied bargaining game with the utility pair $\left(v(H), F(H)\right)$, where $F(H) = \min_{x\in H} \frac{\partial H}{\partial x}$; concave game induces utility pair with $F(H) = \max_{x\in H} \frac{\partial H}{\partial x}$. Here, the coalition assumes the role of Player no. 1 with the characteristic function $v(H)$. The coalition moderator, the Player no. 2, expects the award $F(H)$.

**Proposition**. *The solution* $f(\boldsymbol{S},\varnothing)$ *of a Nash's Bargaining Problem* $\langle \boldsymbol{S},\varnothing \rangle$, *which accompanies a convex (concave) coalition game with characteristic function* $v$, *lies on its Pareto frontier* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *maximizing (minimizing) the product* $v(\Gamma_j)^{\theta} \cdot \frac{\partial \Gamma_j}{\partial \alpha}^{1-\theta}$ *for some* $j = 0,1,\ldots,p$, *and* $0 \le \theta \le 1$.

*Proof:* This statement is an obvious corollary from the Frontier Theorem. ■

In accordance with the basic monotonic property, see above, given some monotonic function $\pi(x;H) \equiv \frac{\partial H}{\partial x}$ on $N \times 2^N$, it is not immediately apparent that there exists some characteristic function $\nu(H)$ for which the function $\pi(x;H)$ constitutes a monotonic marginal utility $\frac{\partial H}{\partial x}$. The following theorem, accommodated in line with the work of Muchnik and Shvartser (1987), addresses this issue.

**The existence theorem**. *For the function* $\pi(x,H)$ *to represent a monotonic marginal utility* $\frac{\partial H}{\partial x}$ *of some supermodular (submodular) function* $\nu(H)$, *it is necessary and sufficient that*

$$\frac{\partial}{\partial y}\frac{\partial H}{\partial x} \equiv \pi(x;H) - \pi(x;H-y) = \pi(y;H) - \pi(y;H-x) \equiv \frac{\partial}{\partial x}\frac{\partial H}{\partial y}$$

*holds for* $x, y \in H \subseteq N$. The interpretation of this condition is left for the reader.

## 7. Heuristic Interpretation

Only the last issue is relevant to our bargaining solution $\Gamma = f(\boldsymbol{S},\varnothing)$ to the supermodular bargaining game. The coalition $\Gamma$ is a stable point with reference to the threshold value $u = F(\Gamma) = \min_{x\in K} \frac{\partial \Gamma}{\partial x}$. This coalition guarantees a gain $u = F(\Gamma)$ to Player no. 2. Therefore, Player no. 2 can prevent anyone $x \notin \Gamma$ outside the coalition $\Gamma \in \boldsymbol{S}$ from becoming a new member of the coalition because the outsider's marginal contribution $\frac{\partial \Gamma}{\partial x}$ reduces the gain guaranteed to Player no. 2. The same incentive governing the behavior of Player no. 2 will prevent some members $x \in \Gamma$ from leaving the coalition. The unconventional interpretation given below might help elucidate this situation.

Let us observe a family of functions on $N \times 2^N$ monotonic towards the second set variable $H$, $H \in 2^N$. Let it be a function $\pi(x;H) \equiv \frac{\partial H}{\partial x}$. We already cited Shapley (1971), who introduced the convex games, with the marginal utility $\frac{\partial H}{\partial x} = \nu(H) - \nu(H-x)$, which is the one of many exact utilizations of the monotonicity $\pi(x,L) \le \pi(x,G)$ for $x \in L \subseteq G$. Authors of some extant studies, including this researcher, refer to these marginal $\nu(H)-\nu(H-x)$ set functions as the derivatives of supermodular functions $\nu(H)$. By inverting the inequalities, we obtain submodular set functions.

Convex coalition game, referring to Shapley(1971) once again, can have a "snowballing" or "band-wagon" effect of cooperative rationality; i.e. in a supermodular game, the cooperative rationality suppresses the individual rationality. In contrast, in submodular games with the inverse property $\pi(x,L)\ge\pi(x,G)$ (an extrapolation this time), the individual rationality suppresses the collective rationality. Hence, it is not beneficial in either case. On a positive note, if the moderator is in charge for coalition formation, the moderator award will be equal to the least marginal utility $u = F(H) = \min_{x\in H}\frac{\partial H}{\partial x}$ of some weakest player in the coalition $H$ under formation. Now, we can focus on a two-person cooperative drama to be played out between the moderator and the coalition.

We start this discussion with our heuristic interpretation. Following the apparatus of monotonic systems in terms of data mining (Mullat, 1971), it is reasonable to find the Pareto frontier in terms of the game theory as well. The potential moderator's bargaining strategy is presented next. First, in the grand coalition $N\equiv\Gamma_0$, the moderator identifies the players with the least marginal utility $u_0 = F(N) = \min_{x\in N}\frac{\partial N}{\partial x}$. The moderator will advise them to stay in line and wait for their awards. All players that have joined the line will be temporarily disregarded in any coalition formation. Following the game convexity, one of the remaining players (i.e. those still remaining in the coalition formation process) must find themselves worse off owing to the players in line being excluded from the process. Moderator would thus suggest to these players to also join the line and wait for their awards. The moderator continues the line construction in the same vein. This process will result in a scenario in which all remaining players $\Gamma_1$ (outside the line) are better off than $u_0$, i.e. better off than those waiting in line for their awards. Now, the moderator repeats the entire procedure upon players $\Gamma_1, \Gamma_2, \ldots$ until all players from $N$ are assigned to wait in line to obtain their awards. Moderator, certainly, keeps a record of the events $0, 1, \ldots$ and is aware when the marginal utility thresholds increases from $u_0$ to $u_1$, etc. It is obvious that the increments are always positive: $u_0 < u_1 < \ldots < u_p$.

What is the outcome of this process? Players staying in line arrange a nested sequence of coalitions $\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \rangle$. The most powerful marginal players, those present when the last event $p$ occurs, form a coalition $\Gamma_p$. The next powerful coalition will be $\Gamma_{p-1}$, etc., coming back once again to the starting event $0$, when the players arrange the grand coalition $N = \Gamma_0$. Our Frontier Theorem guarantees that such a moderator bargaining strategy, in convex games, classifies a Pareto frontier $\langle (\nu(\Gamma_0), u_0), (\nu(\Gamma_1), u_1), \ldots, (\nu(\Gamma_p), u_p) \rangle$ for a bargaining game between the moderator and coalitions under formation.[11] Thus, the game ends when a bargaining agreement is reached between the moderator and the coalition. However, some players might still stay in line, waiting in vain for their awards, because the moderator might not agree to allow them to partake in coalition formation. Indeed, due to the existence of those marginal players, the moderator may lose a large portion of his/her award $F(\Gamma_k)$, for some $k\text{'s} \in \langle 1, \ldots, p \rangle$. [12]

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[11] This sequence of players/elements in line arranges so-called defining sequence in data mining process.

[12] We refer to similar behaviour of players in "Left- and Right-Wing Political Power Design: The Dilemma of Welfare Policy with Low-Income Relief" as political parties bargaining game agents registered under the social security administration.

## 8. CONCLUSION

Nash bargaining solution being understood as a point on the Pareto frontier in Monotonic System might be an acceptable convention in the framework of "fast" calculation. The corresponding algorithm for finding the solution is characterized by a relatively few operations and can be implemented using known computer programming "recursive techniques" on tables. From a purely theoretical perspective, we believe that our technique is a valuable addition to the repertoire presently at the disposal of the game theoreticians. However, our bargaining solution is presently not fully grounded in validated scientific facts established in game theory. Consultations with specialists in the field are thus necessary to develop our work further. In our view, our coalition formations games are sufficiently clear and do not require specific economic interpretations. Nevertheless, they need to be confirmed by other fundamental studies.

## APPENDIX. Illustration of a club formation bargaining game with neither supermodular nor submodular characteristic function.

Recall the health club formation game from Section 2. Given the characteristic function $\nu(H)$, although whether the club members actually arrive at individual payoffs or not is irrelevant, the club formation is still of our interest. Let the game participants $N = \{1,2,3,4,5,6,7\}$ try to organize a club. Let the characteristic (revenue) function comply with the promises of individual employees to participate in the offered health activities in accordance with their survey responses, see Table 1. However, we demand that all five-health activities be materialized.

Define $\nu(H) = |H| + \sum_{x \in H} \sum_{j=1}^{5} a_{x\,j}$, where $H \subseteq N = \{1,2,3,4,5,6,7\}$.

In other words, a promise fulfilled by the club member contributes a Bank Note to the player. In addition to all the promises fulfilled, a side payment per capita is available. According to this rule $\nu(\{1\})=3$, $\nu(\{2\})=5$,… Nonetheless, we are going to change the side payments rule, so that the game transforms into neither supermodular nor submodular game. Note that $\sum_{i}^{7}\nu(\{i\})=\nu(N)=\nu(\{1,2,3,4,5,6,7\})=29$, which renders non-essential game.

Yes, indeed, the employees, whether they choose to cooperate or not, will be discouraged from forming a club arriving at the same gains. To change the situation into that similar to "*the real life cacophonous*", let the side payment per capita be removed for single and pairwise players while keeping the awards intact for all other coalitions for which the size exceeds 2. Thus $\nu(\{1\})=2$, $\nu(\{2\})=4$, $\nu(\{1,2\})=6$, $\nu(\{3,6\})=5$, $\nu(\{2,3,5\})=12$, etc. Moderator's gain, which was defined as $F(H)=\min_{x\in H}\frac{\partial H}{\partial x}\equiv(\nu(H)-\nu(H-x)$, see above, makes the employees' "cooperative behavior" close to grand coalition less profitable for the moderator.

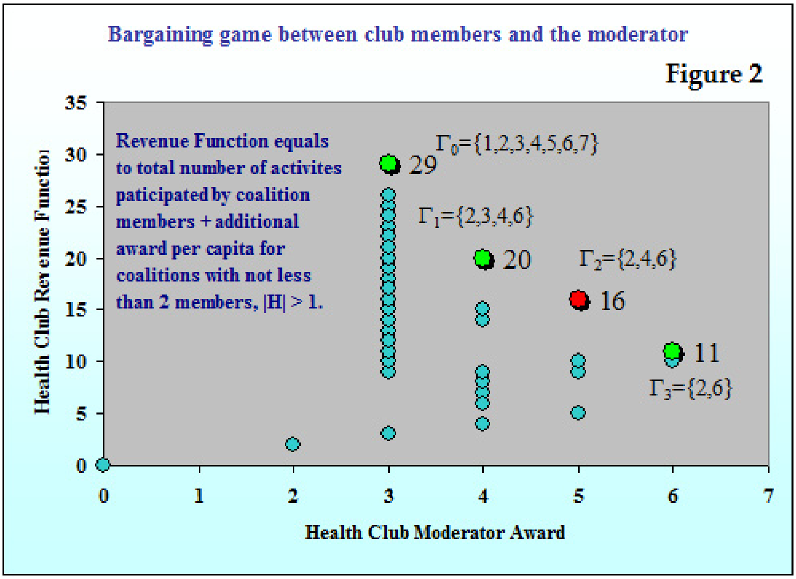
Therefore, the moderator would benefit from encouraging the employees to enter the club of a "reasonable size". In Table 8, we examine this phenomenon using different moderator gain $F(H)$ values.

Boolean Tables

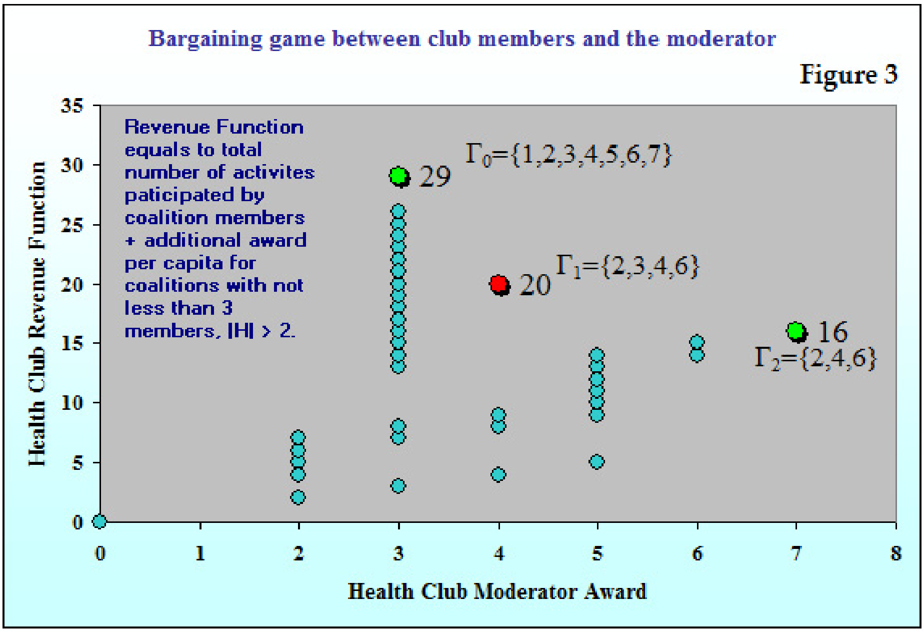
Table 8.

| Health Clubs List | | | | | | | Marginal Utilities p/capita | | | | | | | x | y |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ν(H) | F(H) |
| * | | | | | | | 2 | | | | | | | 2 | 2 |
| | * | | | | | | | 4 | | | | | | 4 | 4 |
| * | * | | | | | | 2 | 4 | | | | | | 6 | 2 |
| | | * | | | | | | | 3 | | | | | 3 | 3 |
| * | | * | | | | | 2 | | 3 | | | | | 5 | 2 |
| - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| | | * | | * | | | | | 3 | | 2 | | | 5 | 2 |
| * | | * | | * | | | 5 | | 6 | | 5 | | | 10 | 5 |
| | * | * | | * | | | | 7 | 6 | | 5 | | | 12 | 5 |
| * | * | * | | * | | | 3 | 5 | 4 | | 3 | | | 15 | 3 |
| | | | * | * | | | | | | 4 | 2 | | | 6 | 2 |
| * | | | * | * | | | 5 | | | 7 | 5 | | | 11 | 5 |
| - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| | . | * | * | * | * | * | | . | 4 | 5 | 3 | 6 | 3 | 21 | 3 |
| * | . | * | * | * | * | * | 3 | . | 4 | 5 | 3 | 6 | 3 | 24 | 3 |
| . | * | * | * | * | * | * | . | 5 | 4 | 5 | 3 | 6 | 3 | 26 | 3 |
| * | * | * | * | * | * | * | 3 | 5 | 4 | 5 | 3 | 6 | 3 | 29 | 3 |

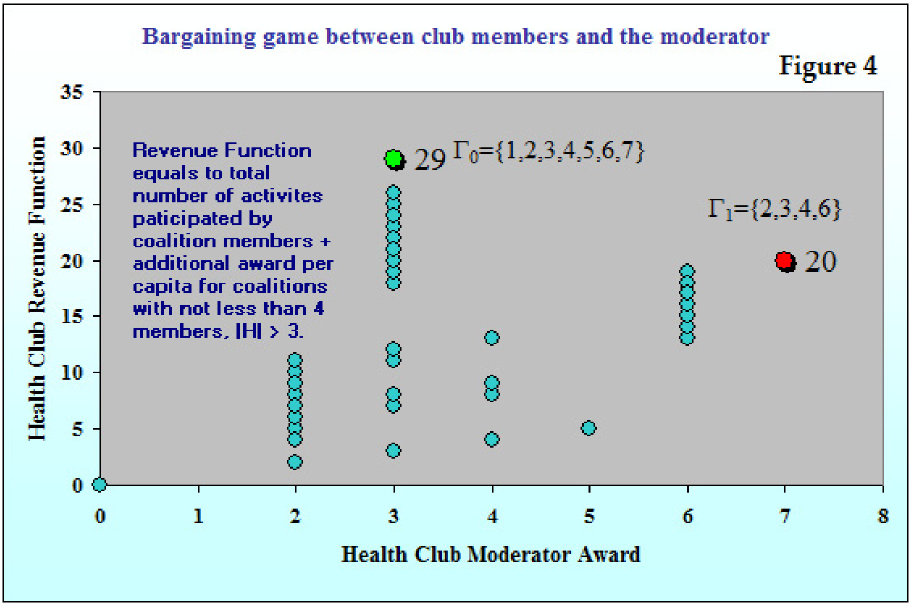
At last, we illustrate the bargaining game in the graph below and make some comments.



N.B. Observe that utility pairs $(29,3)$, $(20,4)$, $(16,5)$ and $(11,6)$ constitute the Pareto frontier of bargaining solutions for the bargaining problem involving the moderator as Bargainer no. 1 and coalitions as Bargainer no. 2. Accordingly, given the grand coalition $N = \Gamma_0 = \{1,2,3,4,5,6,7\}$, three proper coalitions $\Gamma_1 = \{2,3,4,6\}$, $\Gamma_2 = \{2,4,6\}$ and $\Gamma_3 = \{2,6\}$ exist. Solutions $(\nu(\Gamma_1) = 20, F(\Gamma_1) = 4)$ and $(\nu(\Gamma_2) = 16, F(\Gamma_2) = 5)$ maximize the product of players' gains over the disagreement point $(0,0)$ at $20 \cdot 4 = 16 \cdot 5 = 80$. More specifically, as noted at the beginning of the paper, the solution might not be unique and some external considerations may help. For example, the sponsor expenses for $(20,4)$ are equal to $24$, while those pertaining to $(16,5)$ are equal to $21$, which might be decisive. That is the case when the bargaining power $\theta = ½$ of the coalitions $\Gamma_1$, $\Gamma_2$ and the moderator are in balance. Otherwise, choosing the coalition bargaining power $\theta < ½$, the moderator will be better off materializing the solution $(5,16)$. Conversely, coalition $\Gamma_2$ will be better off if $\theta > ½$.



NB. Comparison with Fig. 2 reveals that coalition $\Gamma_3 = \{2,6\}$ is no longer located on the Pareto frontier.




N.B. Comparison with Fig. 3 indicates that coalition $\Gamma_2 = \{2,4,6\}$ no longer lies on the Pareto frontier.

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