Monotonic Systems and Their Properties

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Abstract

This paper presents, for the first time, a systematic description of a new method of structural analysis of data, based on so-called monotonic systems. It is aimed at constructing a systematic description of the data. In this theory, the structure of the data is described as a sequence of mathematical objects - quasi-kernels, which gives this approach a certain resemblance to a classical agglomerative methods (however here the starting point of the sequence - the kernel - is determined by the theory). The paper mainly concerns itself with presenting the mathematical basis for the suggested methods.

1 Problems of Structuring Social-Economic Information and Theory of Monotonic Systems

A number of new approaches have appeared in the area of development of mathematical methods for structural analysis of socio-economic information. The first of these is the use of data classification methods for determining the structure of the data, although other methods [7] are also used. Another aspect is the use of the matrix of pair-wise similarity coefficients between elements as the information basis for classification [5].

Finally, it should be noted that the problem of classification, formulated as some problem of combinatorial optimization, in practically interesting cases can only be solved through a rude approximation, because exact algorithms require exhaustive search. Often there exists some prior information that leads to good initial conditions, or the data has a "nice structure". In such cases a simple approximation algorithm may find an acceptable solution after only a few trials.

In practice, a complicating factor in the analysis of economic and sociological data is a presence of some free "structural" parameter in the formal description of the algorithm, such as a number of classes, or a threshold of similarity that determines a maximally acceptable radius of a "class" in feature space. The algorithm needs to be tuned based on this parameter.

The above mentioned properties of modern methods for structural analysis of socio-economic data present tough problems to those interested in applications since they usually operate on much wider concepts.

Most developers of methods for structural analysis treat these properties of modern systems as characteristics of the problems and therefore put little effort into revising them, or, if they do, then only for very special cases [1]. The need for such a revision, however, is acutely felt in applied studies where specific classification methods receive an increasing amount of attention [3].

Those in the applied sciences use increasingly complicated combinatorial methods to overcome the limitations of methods based only on pair-wise similarity [5]. At the same time, a lot of effort is spent on developing specialized methods for structural analysis of specific types of data that allow one to find exact solutions to the structural analysis problem [2]. Thus arises the problem of developing a universal mathematical apparatus for removing the above-mentioned limitations on the general level of problem statement.

We think that one possible approach to solving this problem lies in intensive use of the theory of monotonic systems [6]. The description of the central mathematical construct of this theory leads to the treatment of observations not as isolated objects or properties but as elements of a system that interact with each other and therefore are important not only in themselves but also with respect to their position relative to other data. As a result, as will be demonstrated in the remainder of this paper, the theory of monotonic systems creates not a single procedure for structural analysis of data but, in a sense, provides a complete and multi-faceted description of the structure of the data.

In the Section 2 we give the exact definition of a monotonic system. In this context we also introduce the concept of "central structure of the data" - the kernel, that can be viewed as a system of class "centers". At the same time, a method for finding the exact solution of the corresponding optimization problem is found. The main structural parameter, the number of centers, is not given in advance but is determined in the process of solving the problem. The solution has a number of properties that are important for interpreting the results and are discussed in that same section.

The third section expands the concept of the central structure (kernel) to the level of complete structural description of the data. Such complete structure consists of a sequence of quasi-kernels where each element can be seen as a specific (and automatically determined!) level of of detail in the interpretation of the data. This sequence as a whole determines, in some sense, a "scope" of data.

The last section is devoted to describing the procedures for finding such a sequence. It is shown that the complexity of such procedure does not exceed N^3, where N is the number of observations in data. Furthermore, the complexity of the procedure can be reduced if only the "core" part of the structure, rather than the whole, is of interest.

The theory of monotonic systems determines some new class of combinatorial optimization problems that can be be effectively solved and therefore has greater importance than just another on method for processing empirical data. In particular, it can be applied to problems of mathematical modeling of organizational structures, structural distribution systems, formation of interactive coalition and many others [4].

2 Extremal Subsystems of Monotonic Systems

Consider a finite set W, $|W| = N$. Let π be a scalar function defined on all pairs (i, H) where $H \subseteq W$ is an arbitrary subset of W, and $i \in H$ is its element. $\pi(i, H)$ can be interpreted as a measure of closeness of an element i to H, "importance" or "weight" of i in H, degree of influence of i on H and so on. If we define a distance matrix $||\rho_{ij}||$ on all pairs of elements of set W, then we can define $\pi(i, H)$ as a sum of distances from element i to all other elements of subset H of W:

$$\pi(i, H) = \textstyle\sum_{j \in H} \rho_{ij}.$$

We shall refer to $\pi(i, H)$ as a weight of element i in set H.

Consider a system $< W, \pi >$, consisting of elements of a finite set W together with function $\pi(i, H)$ defined on a set of all pairs (i, H), $H \subseteq W$, $i \in H$. A system $< W, \pi >$ is called monotonic if

$$\pi(i, H \backslash j) \leq \pi(i, H), \forall i, j \in H, i \neq j, \forall H \subseteq W, \tag{1}$$

or

$$\pi(i, H \backslash j) \geq \pi(i, H), \forall i, j \in H, i \neq j, \forall H \subseteq W, \tag{2}$$

Systems for which (1) holds are called $\ominus$-monotonic and are denoted by $< W, \pi^- >$. Systems for which (2) holds are called $\oplus$-monotonic and are denoted by $< W, \pi^+ >$.

Definition: $\ominus$-monotonic system $< W, \pi^- >$ is a finite set W together with a scalar function $\pi(i, H)$, (i, H), $H \subseteq W$, $i \in H$, that satisfies property (1).

$\oplus$-monotonic system is defined analogously. Inequality (2) in this case also expresses the monotonic property of a system. Thus we obtain monotonic systems of two types (or "two signs").

The central task of the theory of monotonic systems is extraction of a certain extremal subsystem of a monotonic system, also known as the defining set (or the maximum kernel). This task can be formally stated and solved as described in the remaining part of this paper.

A scalar function F^- is defined on a set of all subsets H of W in a $\ominus$-monotonic system so that for any $H \subseteq W$:

$$F^-(H) = \min_{i \in H} \pi^-(i, H), \forall H \subseteq W \tag{3}$$

Definition: A kernel (extremal subsystem) of $\ominus$-monotonic system $< W, \pi^- >$ is a subset of W where the function F^- attains its maximum value.

Analogously we can define a kernel of $\oplus$-monotonic system through the introduction of a function $F^+(H)$:

$$F^+(H) = \max_{i \in H} \pi^+(i, H), \forall H \subseteq W \tag{4}$$

that attains its minimum value on such kernels.

In what follows, $F(H)$ and $\pi(i, H)$ will be used whenever it is clear from the context whether $\ominus$-monotonic or $\oplus$-monotonic system is discussed, or when the discussion pertains to both types of systems. The same holds for other notation.

Let $A =< \alpha_1, \ldots, \alpha_N >$ be an arbitrarily ordered sequence of elements of W, and define $\overline{H}(A)$, or simply $\overline{H}$, as $\overline{H} =< H_1, \ldots, H_N >$, a sequence of nested subsets of W, where $H_1 = W, H_2 = H_1 \backslash \alpha_1, \ldots, H_{k+1} = H_k \backslash \alpha_k, \ldots, H_N = \alpha_N$.

Definition: An ordered sequence A of elements of W is called a defining sequence of a $\ominus$-monotonic system $< W, \pi^- >$ if the corresponding sequence of subsets $\overline{H}$ contains a subsequence $\overline{\Gamma} =< \Gamma_1, \ldots, \Gamma_p >$, where $\Gamma_1 = H_1 = W$ and

$$\pi(\alpha_k, H_k) < F(\Gamma_{j+1}), \forall \alpha_k \in \Gamma_j \backslash \Gamma_{j+1}, j = \overline{1, p-1} \tag{5}$$

$$F(L) \leq F(\Gamma_p), \forall L \subset \Gamma_p. \tag{6}$$

A defining sequence of $\oplus$-monotonic system is described analogously, except for inequalities (5) and (6) that are replaced by:

$$\pi(\alpha_k, H_k) > F(\Gamma_{j+1}), \forall \alpha_k \in \Gamma_j \backslash \Gamma_{j+1}, j = \overline{1, p-1} \tag{7}$$

$$F(L) \geq F(\Gamma_p), \forall L \subset \Gamma_p. \tag{8}$$

Definition: Set G, $G \subseteq W$, is called a defining set of a monotonic system $< W, \pi >$ if there exists a defining sequence for which $\Gamma_p = G$.

We shall denote a defining set of a $\ominus$-monotonic system by G^-, and that of a $\oplus$-monotonic system - by G^+. The following are the two central theorems[1]of monotonic system theory [6].

Theorem 1 *Function $F^-(H)$ attains its global maximum on a defining set G^- of $\ominus$-monotonic system. Defining set is unique. All sets where $F^-(H)$ attains its global maximum, i.e. all kernels of $\ominus$-monotonic system, lie inside the defining set*

Let X be a set of all subsets of W.

Theorem 2 *System of sets of X where F^- attains the global maximum is closed under binary union operation.*

Theorems 1 and 2 for $\oplus$-monotonic systems are stated analogously, with maximum replaced by minimum.

Consider a numerical sequence $\overline{\epsilon} =< \epsilon_1, \ldots, \epsilon_p >$, where $\epsilon_j = F(\Gamma_j)$, $j = \overline{1, p}$.

Lemma 1 *For $\ominus$-monotonic system the following chain of inequalities holds:*

$$\epsilon_1 < \epsilon_2 < \ldots < \epsilon_p, \tag{9}$$

while for $\oplus$-monotonic system

$$\epsilon_1 > \epsilon_2 > \ldots > \epsilon_p, \tag{10}$$

[1]Since most properties (facts) concerning monotonic systems are symmetric for $\ominus$- and $\oplus$-monotonic systems, theorems stating those properties will be marked with "-" or "+". The sign is omitted when referring to a theorem if it is clear from the context what type of system is being discussed.

Proof: Let γ_j be the first element of Γ_j ($j = \overline{1, p-1}$) in a defining sequence. Since, by definition, sequence $\overline{\Gamma}$ is a subsequence of sequence $\overline{H}$, it follows that for some k $\Gamma_j = H_k \in \overline{H}$ and $\gamma_j = \alpha_k$. Then, according to property (5) of defining sequence and definition (3) of the function F, we have:

$$\epsilon_k = F(\Gamma_j) \leq \pi(\gamma_j, \Gamma_j) = \pi(\alpha_k, H_k) < F(\Gamma_{j+1}), j = \overline{1, p-1},$$

which implies (9). The chain of inequalities in (10) is proved analogously. End of Proof.

Proof of Theorem 1: Assume that the defining set G exists [2]. We need to prove that $F(G) \leq F(H), \forall H \subseteq W$.

Assume that there exists a set $L \subseteq W$, such that

$$F(G) \leq F(L). \tag{11}$$

Then there are two possibilities: either $L \subseteq G$ or $L \backslash G \neq \emptyset$. Consider the first case. By definition of G there exists a defining sequence A of elements of W with property (6), that is the strict inequality $F(G) < F(L)$ cannot hold and therefore only the equality holds in (11). In that case, the first and the third statements of the theorem are proven. The uniqueness of G will be proven later for both cases simultaneously.

Consider the second possibility ($L \backslash G \neq \emptyset$). Let H_n be the minimal set of sequence $\overline{H}$ corresponding to the defining sequence A, containing the set $L \backslash G$. That is, there exists $\alpha_n \in H_n$, but $\alpha_n \notin H_{n+1}$. Now, let Γ_S be the minimal set of sequence $\overline{\Gamma}$ such that $H_n \subseteq \Gamma_S$ but $H_n \nsubseteq \Gamma_{S+1}$. This means that $\alpha_n \in \Gamma_S$,$L \subseteq \Gamma_S$ (since $L \subseteq H_n \subseteq \Gamma_S$), $\alpha_n \notin \Gamma_{S+1}$.

Then, based on property (5) of defining sequence and Lemma 1, we can conclude that

$$\pi(\alpha_n, H_n) < F(\Gamma_{S+1}) < F(G) \tag{12}$$

It follows from (11) and (12) that

$$\pi(\alpha_n, H_n) < F(G) < F(L).$$

But according to monotonicity property (since $L \subseteq H$)

[2]The existence of the defining set and defining sequence is not proven here. Section 4 contains construction of the defining sequence and thereby proves the existence of the defining set. Thus a complete proof of the theorem 1 consists of the proof given here combined with algorithms in Section 4.

$$\pi(\alpha_n, L) \leq \pi(\alpha_n, H_n),$$

implying that

$$\pi(\alpha_n, L) \leq F(L) = \min_{i \in L} \pi(i, L).$$

The above inequality implies that set L contains an element α_n with the weight that is strictly less than the minimum, which is impossible. Therefore it follows that L can be only a subset of G, i.e. all sets where F attains its global maximum are contained inside G. It remains to show that if the defining set exists, it must be unique. From the above discussion, if there is a defining set $G' \neq G$, then it must be included in G. But similarly, G must be included into G', thus proving uniqueness of G.

Proof of Theorem 2: Let G_1 and G_2 be two different kernels of $\ominus$-monotonic system. We need to show that $G_1 \cup G_2$ is also a kernel, i. e. that

$$F(G_1 \cup G_2) = F(G_1) = F(G_2) = \max_{H \subseteq W} F(H). \tag{13}$$

Since $F(H)$ attains maximum on G_1 and G_2, it follows that

$$F(G_1 \cup G_2) \leq F(G_1), F(G_1 \cup G_2) \leq F(G_2). \tag{14}$$

On the other hand, let $g \in G_1 \cup G_2$ be an element such that $\pi(g, G_1 \cup G_2) = F(G_1 \cup G_2)$. Assume $g \in G_1$. (if $g \in G_2$ the argument proceeds similarly). Then, by monotonicity property we have

$$\pi(g, G_1) \leq \pi(g, G_1 \cup G_2),$$

and, therefore,

$$F(G_1) = \min_{i \in G_1} \pi(i, G_1) \leq \pi(g, G_1 \cup G_2) = F(G1 \cup G_2). \tag{15}$$

From (14) and (15) we obtain (13), thereby proving the theorem.

Theorems 1 and 2 establish important structural properties of monotonic systems: the set of all of its kernels forms a closed system under the union operation, and the union of all kernels is the defining set and the largest kernel.

An algorithm for constructing defining sequence that will be discussed in Section 4 allows effective (without exhaustive search) extraction (according to the definition) of the largest kernel, to which we shall below refer as simply the kernel. In other words, the kernel of a monotonic system is the last set $G = \Gamma_p$ in the sequence $\overline{\Gamma}$ that is fixed in the process of constructing the defining sequence.

Thus, the kernel of monotonic system can be determined by constructing a defining sequence of its elements. In connection to this we examine the set of all possible defining sequences of a monotonic sequence.

Generally speaking there can be many distinct defining sequences for one monotonic system that satisfy the definition. However, theorems 1 and 2 demonstrate that a monotonic system has only one kernel, i.e. the largest set where function $F(H)$ attains its global extremum value. Therefore all defining sequences are structured in such a way that the last set G in the corresponding sequences $\overline{\Gamma}$ is the same. This fact limits the number of defining sequences of a monotonic sequence that need to be examined to a given single defining sequence.

Indeed, assume that a way for transforming given defining sequence into another defining sequence is known. Then a possible method for enumerating all defining sequences is to use a constructive approach for finding one defining sequence and then generating the set of all defining sequence from the first one using this transformation.

The following theorem demonstrates bounds on a set of all defining sequences that can be constructed via such a method.

Theorem 3 *If α_S and α_t, $S < t$ are two elements of a defining sequence A such that $\alpha_S, \alpha_t \in G$, then the sequence*

$$A' =< \alpha_1, \dots, \alpha_{S-1}, \alpha_t, \alpha_{S+1}, \dots, \alpha_{t-1}, \alpha_S, \alpha_{t+1}, \dots, \alpha_N >$$

is also a defining sequence. However if $\alpha_S \in W \backslash G, \alpha_t \in G$, then A' is not a defining sequence.

Proof: Let us start by proving the first statement of the theorem. Let $\alpha_S, \alpha_t \in G$. Since $G = \Gamma_p$ is the last set in sequence $\overline{\Gamma}$ (and m-th set in sequence H), sequences A and A', H and H' are identical up to the element i_{m-1} and set H_{m-1} accordingly. Therefore sequences $\overline{\Gamma}$ and $\overline{\Gamma}'$ also coincide, and condition (5), which by definition holds for $\overline{\Gamma}$, also holds for $\overline{\Gamma}'$. Condition (6) clearly holds regardless of the ordering of elements inside G.

The second statement of the theorem concerns the situation where $\alpha_S \in W \backslash G, \alpha_t \in G$. Taking into consideration the fact that $\alpha_S{}' = \alpha_t \in G$ and $G \in H_S{}' = H_S$ and using the definition of the function $F(H)$ and monotonicity property, we obtain the following:

$$F(G) \leq \pi(\alpha_t, G) \leq \pi(\alpha_t, H_s) = \pi(\alpha_S, H_S{}').$$

This contradicts property (5) of defining sequence with respect to sequence A'. The proof is complete.

At the same time we have proven that in the second case (when $\alpha_S \in W \backslash G, \alpha_t \in G$) set $(G \cup \alpha_S) \backslash \alpha_t$ is not the kernel.

The theorem 3 does not cover a third possibility, that of $\alpha_S, \alpha_t \in W \backslash G$. It can be shown that the rearrangement of arbitrary elements outside the kernel can lead to both preservation and violation of properties of a defining sequence (in particular of property (5)).

Since the set of transformations of defining sequences one into another is in fact the set of all possible permutations of elements of G together with some permutations of elements of $W \backslash G$, the number of all defining sequences of a monotonic system is bounded above by $|W \backslash G|! \times |G|!$.

3 General Properties of Monotonic Systems

In this section we shall examine the common properties of $\oplus$- and $\ominus$-monotonic systems. Therefore when these properties for $\oplus$- and $\ominus$-monotonic systems are symmetric, only the properties of $\ominus$-monotonic systems shall be stated. Analogous properties for $\oplus$-monotonic systems can be derived by substituting inequality signs $>$ and $<$, $\geq$ and $\leq$, max and min, etc.

The main purpose of this section is to demonstrate that any set $\Gamma_j, j = \overline{1, p-1}$ from sequence $\overline{\Gamma}$ possesses certain extremal properties, analogous to the properties of the kernel $G = \Gamma_p$. This will also indicate the importance of the elements of the sequence $\overline{\Gamma}$ for the problems of structural analysis, in addition to that of the kernel G itself.

Firstly, we give another, more precise, definition of kernel.

Definition: A kernel of a $\ominus$-monotonic system $< W, \pi >$ is the largest subset G of set W where function $F(H)$ attains its maximum. That is, G must satisfy the following properties:

$$F(G) = \max_{H \subseteq W} F(H) \tag{16}$$

$$|G| = \max_{H \subseteq W, F(H) = F(G)} |H| \tag{17}$$

Properties (16) and (17) can be restated in a different form as:

$$F(H) < F(G), \forall H \subseteq W, |H| > |G| \tag{18}$$

$$F(H) \leq F(G), \forall H \subseteq W, |H| \leq |G| \tag{19}$$

On the other hand, we can state, based on Theorems 1 and 2, that for the defining set G the following properties hold:

$$F(H) < F(G), \forall H \subseteq W, H \backslash G \neq \emptyset \tag{20}$$

$$F(H) \leq F(G), \forall H \subseteq W, H \subseteq G. \tag{21}$$

It can be demonstrated that relations (18), (19) follow from relations (20), (21). Indeed, the set of all subsets, including single elements not belonging to G, that is the set $\{H : H \backslash G \neq \emptyset\}$, consists of two parts: set of subsets of W with more elements that in G, and set of subsets of W for which simultaneously $|H| \leq |G|$ and $H \backslash G \neq \emptyset$. The second part is at the same time a part of $\{H : |H| \leq |G|\}$, where the remaining part is the set $\{H : H \subseteq G\}$. Combining these relations leads to the desired statement.

This in turn means that the Theorem 1 from the previous section can be restated as follows:

Theorem 4 *The defining set is the unique kernel.*

In other words, there can be no other maximal kernels except for the defining set. Therefore, the concepts of the defining set and of the kernel that were previously defined independently refer to the same object.

The kernel therefore can be equivalently defined as:

- the defining set
- the set satisfying relations (16) and (17), or (18) and (19), or (20) and (21)

These different definitions will be used in what follows for proving properties of monotonic systems. Additionally, it is natural to attempt to find a definition of kernel that would impose least strict and easily checked constraints on a given set. In such circumstances the proof of a fact that some set is the kernel of a monotonic system would be straightforward. In particular, in order to show the correctness of an algorithm for extracting the kernel of a monotonic system that is described in the next section, it is useful to introduce another definition of the kernel.

Let G be a set satisfying the following relations:

$$F(H) < F(G), \forall H \subseteq W, H \supset G \tag{22}$$

$$F(H) \leq F(G), \forall H \subseteq W, H \subseteq G. \tag{23}$$

Theorem 5 *The kernel G of $\ominus$-monotonic system satisfies relations (22) and (23). A set G, satisfying relations (22) and (23), is the kernel.*

Proof: We have to prove that relations (16) and (17), or (18) and (19), or (20) and (21) imply relations (22) and (23), and the other way around. It is easy to see that (21) and (23) coincide, while (22) follows from (20) or (16). That is, (22) and (23) hold for the kernel. Let us now prove that (22) and (23) imply (20) and (21).

Let H, $H \subseteq W, H \backslash G \neq \emptyset$. Consider set $H \cup G$. Since $H \cup G \supset G$, then by (22) we have

$$F(H \cup G) < F(G).$$

Let α be an element of $H \cup G$ s.t

$$F(H \cup G) = \pi(\alpha, H \cup G).$$

Suppose $\alpha \in G$. Then, using monotonicity property and the definition of $F(H)$, we obtain the following sequence of inequalities:

$$F(G) \leq \pi(\alpha, G) \leq \pi(\alpha, H \cup G) = F(H \cup G) < F(G)$$

that leads to a contradiction. Now, suppose $\alpha \in H \backslash G$. In this case we obtain an analogous sequence of inequalities:

$$F(H) \leq \pi(\alpha, H) \leq \pi(\alpha, H \cup G) = F(H \cup G) < F(G),$$

that proves (20) and therefore (16). The proof is complete.

We have proven that (22) and (23) provide an equivalent definition of the kernel. At the same time it is easy to see that proving (22) and (23) for some set G requires examining a smaller number of subsets H that proving of relations (18), (19) or (20), (21).

In addition, for further exploration of properties of monotonic systems we will need a narrower class of defining sequences that arise with adding another condition to the ones already given ((5) and (6)) in the previous definition (in Section 2).

Definition: An ordered sequence $A =< \alpha_1, \ldots, \alpha_N >$ of all elements of W is called a maximal defining sequence if the corresponding sequence of set $\overline{H}$ contains a subsequence $\overline{\Gamma} =< \Gamma_q, \ldots, \Gamma_p >$ where $\Gamma_1 = H_1 = W$ for which, in addition to the properties (5) and (6) the following holds:

$$\pi(\alpha_k, H_k) \leq F(\Gamma_j), \forall \alpha_k \in \Gamma_j \backslash \Gamma_{j+1}, j = \overline{1,p}. \tag{24}$$

The idea behind this definition is that for none of the sets H_i, s.t. $\Gamma_j \supset H_k \supset \Gamma_{j+1}$, property (5) holds. Therefore, no set H_k from sequence $\overline{H}$ can be added to the sequence $\overline{\Gamma}$ between $j-\mu$ and $(j+1)-\mu$ of its elements, that is between Γ_j and Γ_{j+1}. The definition given in Section 2 however allowed such situation.

In the rest of this paper, maximal defining sequence, as defined here, is implied whenever a defining sequence is mentioned.

We now demonstrate that for sets $\Gamma_j, j = \overline{1,p}$ properties analogous to (18)-(23) hold.

Theorem 6 *The following statements are true for the sets Γ_j of sequence $\overline{\Gamma}$:*

$$F(H) < F(\Gamma_j), \forall H \subseteq W, H \backslash \Gamma_j \neq \emptyset, j = \overline{2,p} \tag{25}$$

$$F(H) \leq F(\Gamma_j), \forall H \subseteq \Gamma_j, H \supset \Gamma_{j+1} j = \overline{1,p-1}. \tag{26}$$

The analogy between (20), (21) on the one hand, and (25), (26) on the other is clear. In order to derive even more similar expressions we need the following notation: $\Gamma_0 = \Gamma_1 = W$ and $\Gamma_{p+1} = \emptyset$ ($\Gamma_p = G$ is the last nonempty set in the sequence $\overline{\Gamma}$). Then relations (20), (21) and (25), (26) can be combined as follows:

$$F(H) < F(\Gamma_j), \forall H \subseteq W, H \backslash \Gamma_j \neq \emptyset, j = \overline{1,p} \tag{27}$$

$$F(H) \leq F(\Gamma_j), \forall H \subseteq \Gamma_j, H \supset \Gamma_{j+1}, j = \overline{1,p}. \tag{28}$$

Thus, the difference between the kernel $G = \Gamma_p$ and other sets $\Gamma_j \neq \Gamma_p$ in the sequence $\overline{\Gamma}$ is that the kernel's successor in the sequence is an empty set. Also, as follows from the Lemma 1, $F(G) > F(\Gamma_j) j = \overline{1,p-1}$. In all other respects the extremal properties of the kernel and of any other set in the sequence $\overline{\Gamma}$ are the same.

Corollary 1 *For any subset H of W, s.t $\Gamma_j \supset H \supset \Gamma_{j+1}$ for some $j = \overline{1,p-1}$ the following is true: $F(H) \leq F(\Gamma_j) = \epsilon_j$ $F(H) < F(\Gamma_{j+1}) = \epsilon_{j+1}$*

Proof of Theorem 6: Let us prove (25) by contradiction. Consider set $H \subseteq W$, where

$$F(H) \geq F(\Gamma_j) \tag{29}$$

but not $H \subseteq \Gamma_j$. The last part implies $H \backslash \Gamma \neq \emptyset$. Let α_n be the first element of set H in the defining sequence, i.e. $\alpha_n \in H$, $H_n \supseteq H$, not $H_{n+1} \supseteq H$. Then $H_n \supset \Gamma_j$, that is $H_n \subseteq H \cup \Gamma_j$. Then, by monotonicity property and (5), we have

$$F(H) \le \pi(\alpha_n, H) \le \pi(\alpha_n, H_n) < F(\Gamma_j)$$

contradicting (29). Thus the first statement of the theorem is proven [3].

Relation (26) is proven analogously, except that instead of (29) it is necessary to use the strict inequality $F(H) > F(\Gamma_j)$ where $H \not\subseteq \Gamma_{j+1}$, $H \le \Gamma_j$, and instead of (5) - relation (24). The proof is therefore complete.

The statement of the Theorem 6 become more interpretable if we introduce the following "geometric" notions. *Definition:* An interval $[D, E]$, where $D \supset E$ in the set X of all subsets of W is a family of elements T, $T \in X$, that satisfies the relation:

$$[D, E] = T, D \supseteq T \supseteq E. \tag{30}$$

If a strict inclusion holds in the left or the right parts of relation $D \subseteq T \subseteq E$ then the square bracket is replaced by the circular one in the interval notation:

$$[D, E) = T, D \supseteq T \supset E$$

$$(D, E] = T, D \supset T \supseteq E$$

$$(D, E) = T, D \supset T \supset E$$

Definition: A set L, $L \subseteq W$ is a maximum of a function $F(H)$ on an interval $[D, E]$ (s.t. $L \in [D, E]$ if it satisfies

$$F(L) \ge F(T), \forall T \in [D, E]. \tag{31}$$

Extrema on the open and semi-open intervals are defined analogously. The maximum is called strict if the strict inequality holds in (31).

Theorem 6 can be restated using the above definitions as follows.

Theorem 7 *Each set* $\Gamma_j, j = \overline{1, p}$ *of the sequence* $\overline{\Gamma}$ *is a maximum of the function* $F(H)$ *on the interval* $[\Gamma_1 = W, \Gamma_{j+1})$.

All sets $H, H \subseteq W$ where $F(H)$ has value $F(\Gamma_j)$ for some $j = 1, \dots, p$ lie inside set Γ_j. All sets $H, H \subseteq W$ where $F(H)$ has value greater than $F(\Gamma_j)$ for some $j = 1, \dots, p-1$ lie inside the set Γ_{j+1}. The proof is completely identical to the proof of the Theorem 6. Relations (25), (26) mean exactly

[3] Thus the proof of the first statement of Theorem 6 relies only on relation (5) of the general definition of the defining sequence in Section 2

that the set Γ_j is a maximum of $F(H)$ on the interval $[W, \Gamma_{j+1}]$. Also, the set Γ_j is a strict left-side maximum on the interval $[W, \Gamma_j]$ and a non-strict right-side maximum on the interval $[\Gamma_j, \Gamma_{j+1}]$.

For completeness of description of extremal properties of the sets Γ_j, we introduce two more concepts.

Definition: A neighborhood of set H is the union of all sets L such that $H \subset L \subseteq W$.

In other words, the neighborhood of H is the interval $[W, H]$.

Definition: A set H^0 is a strict local maximum of the function $F(H)$ if it is a strict maximum of this function in its neighborhood.

Corollary 2 *A set H^0 is a strict local maximum of the function $F(H)$ if and only if $H^0 = \Gamma_j$ where $\Gamma_j \in \overline{\Gamma}$.*

So, theorem 7 establishes that the function $F(H)$ has a very simple structure: it has $p \leq N = |W|$ strictly increasing local maxima. The last of these is also a global maximum. It is precisely this simplicity that allows for the construction of a fast algorithm for extracting the kernel: it is only necessary to build a procedure for sequential examination of all the local maxima of $F(H)$.

The analogy between $G = \Gamma_p$ and the other set Γ_j of the sequence $\overline{\Gamma}$ can also be seen at the set-theoretic level. This is established in the following theorem.

Theorem 8 *The system of the subsets of W where F^- attains or exceeds the value of $F(\Gamma_j)$ for some $j = \overline{1,p}$ is closed with respect to the binary union operation.*

Proof: Let E_1, E_2 be two distinct set in W such that

$$F(E_1) \geq F(E_2), F(E_2) \geq F(\Gamma_j), j = \overline{1,p}. \tag{32}$$

According to the theorem 7 $E_1, E_2 \subseteq \Gamma_j$. We need to show that the following inequality is true:

$$F(E_1 \cup E_2) \geq F(\Gamma_j), j = \overline{1,p}. \tag{33}$$

(it is easy to see that $E_1 \cup E_2 \subseteq \Gamma_j$). Let $\alpha \in E_1 \cup E_2$ be an element where $F(E_1 \cup E_2)$ attains its value: $\pi(\alpha, E_1 \cup E_2) = F(E_1 \cup E_2)$. Assume without loss of generality that $\alpha \in E_1$. Then by monotonicity property we have:

$$\pi(\alpha, E_1) \leq \pi(\alpha, E_1 \cup E_2),$$

and therefore

$$F(E_1) = \min_{i \in E_1} \pi(\alpha, E_1) \leq \pi(\alpha, E_1) \leq \pi(\alpha, E_1 \cup E_2) = F(E_1 \cup E_2). \quad (34)$$

Combining (32) and (34) gives (33). The proof is complete [4].

It is easy to notice a strong similarity between theorems 7 and 8 with 1 and 2. Only a statement regarding the uniqueness of the sets Γ_j from sequence $\overline{\Gamma}$ is missing, that is remedied with the following theorem:

Theorem 9 *If A and A' are two distinct maximal defining sequences of a $\ominus$-monotonic system $< W, \pi >$, then the corresponding sequences $\overline{\Gamma}$ and $\overline{\Gamma}'$ coincide:*

$$p = p',\ \Gamma_j = \Gamma'_j, \forall j = \overline{1, p}.$$

Proof: By contradiction. Let $\Gamma_S{}'$ be the first set in the sequence $\overline{\Gamma}'$ such that $\Gamma_S \neq \Gamma'_S$, but $\Gamma_j = \Gamma'_j, \forall j = \overline{1, S-1}$. (Note that $\Gamma_1 = \Gamma'_1 = W$ by definition. Two cases are possible: $\Gamma'_S \subset \Gamma_S$ or $\Gamma'_S \backslash \Gamma_S \neq \emptyset$. We shall consider them one at a time. In the first case, according to the theorem 3, for a set $H = \Gamma_S$ s.t. $\Gamma_{S-1} = \Gamma'_S \supset H \supset \Gamma'_S$ we have $F(\Gamma_S) \leq F(\Gamma_{S-1})$. This inequality contradicts relation (9) for the sequence $\overline{\Gamma}$ (lemma 1). In the second case $\Gamma'_S \backslash \Gamma_S \neq \emptyset$. Let H be the minimal set of sequence $\overline{H}$ such that $H_n \supset \Gamma'_S$. Clearly, $H_n \subseteq \Gamma_{S-1}, \alpha_n \in \Gamma'_S$, where $\alpha_n \in H_n$, $\alpha_n \notin H_{n+1}$. Since $\Gamma_{S-1} = \Gamma'_{S-1}$ by our assumption, we have $F(\Gamma_{S-1}) = F(\Gamma'_{S-1}) < F(\Gamma'_S)$. On the other hand, considering the monotonicity property for sets Γ'_S and H_n, $\Gamma'_S \subset H_n$, the definition of $F(H)$ and corollary to theorem 6, we have:

$$F(\Gamma'_S) \leq \pi(\alpha_n, \Gamma'_S) \leq \pi(\alpha_n, H_n) \leq F(\Gamma_{S-1}).$$

Once again we have arrived at a contradiction with lemma 1. The proof is complete.

According to the theorem 6, the sequence of nested sets $\overline{\Gamma}$ can be seen as a characteristic of the structure of the monotonic system that is invariant with respect to relations and transformations of one defining sequence into another.

[4] It is straightforward to check that theorem 8 is still valid if we use the general definition of a defining sequence given in Section 2.

Consider once again the maximal defining sequence and the corresponding set $\overline{\Gamma} =< \Gamma_1, \ldots, \Gamma_p >$. It should be noted that sets $\Gamma_{j-1} \backslash \Gamma_j$, $j = \overline{2,p}$ also possess certain extremal properties. Specifically, in addition to monotonic system $< W, \pi >$ defined on set W, we can define on $W \backslash \Gamma_j$ another monotonic system $< W \backslash \Gamma_j, \pi' >$, where

$$\pi'(i, H) = \pi(i, H \cup \Gamma_j), \forall H \subseteq (W \backslash \Gamma_j), i \in H.$$

Theorem 10 *The kernel of a monotonic system* $< W \backslash \Gamma_j, \pi' >$ *is the set* $G' = \Gamma_{j-1} \backslash \Gamma_j$.

Proof: We need to show that

$$F'(H) < F'(G), \forall H \subseteq (W \backslash \Gamma_j), H \backslash G' \neq \emptyset$$

$$F'(H) \leq F'(G), \forall H \subseteq (W \backslash \Gamma_j), H \subseteq G'.$$

Let us prove the first of these:

$$F'(H) = \min_{i \in H} \pi'(i, H) = \min_{i \in H} \pi(i, H \cup \Gamma_j).$$

Since

$$\min_{i \in \Gamma_j} \pi(i, \Gamma_j) > \min_{i \in \Gamma_j \cup H} \pi(i, H \cup \Gamma_j), \forall H \subseteq W \backslash \Gamma_j, H \nsubseteq G'$$

and since the function $\pi(i, H)$ is monotonic, it follows that

$$\min_{i \in H} \pi(i, H \cup \Gamma_j) = \min_{i \in H \cup \Gamma_j} \pi(i, H \cup \Gamma_j), \forall H \subseteq W \backslash \Gamma_j, H \not\subset G'.$$

This means that

$$F'(H) = F(H \cup \Gamma_j) < F(\Gamma_{j-1}) = F(G \cup \Gamma_j) =$$

$$= \min_{i \in H \cup \Gamma_j} \pi(i, G' \cup \Gamma_j) = \min_{i \in G'} \pi(i, G' \cup \Gamma_j) = F'(G')$$

Let us now prove the second statement. By using analogous reasoning to the proof of the first statement, we obtain

$$F'(H) = F(H \cup \Gamma_j) \leq F(\Gamma_{j-1}) = F'(G')$$

The proof is complete.

As a conclusion of the comparative exploration of the properties of the kernel $G = \Gamma_p$ and other elements Γ_j of the sequence $\overline{\Gamma}$ we state two more special properties of such sets.

Lemma 2 *For any set $\Gamma_j, j = \overline{2,p}$ of sequence $\overline{\Gamma}$ the following relation holds:*

$$\max_{\alpha_n \in W \setminus \Gamma_j} \pi(\alpha_n, H_n) \leq \max_{\alpha_n \in \Gamma_{j-1} \setminus \Gamma_j} \pi(\alpha_n, H_n) \tag{35}$$

Proof: Since it is clear that $W \setminus \Gamma_j \supset \Gamma_{j-1} \setminus \Gamma_j$ it remains to show that

$$\max_{\alpha_n \in W \setminus \Gamma_{j-1}} \pi(\alpha_n, H_n) \leq \max_{\alpha_n \in \Gamma_{j-1} \setminus \Gamma_j} \pi(\alpha_n, H_n).$$

Indeed, note that the right hand side of this inequality is at least $F(\Gamma_{j-1})$, since the set Γ_{j-1} always contains an element $\gamma_{j-1} \in \Gamma_{j-1} \setminus \Gamma_j$ such that $\pi(\gamma_{j-1}, \Gamma_{j-1}) = F(\Gamma_{j-1})$. Therefore, $\max_{\alpha_n \in \Gamma_{j-1} \setminus \Gamma_j} \pi(\alpha_n, H_n) \geq F(\Gamma_{j-1})$. On the other hand, using the definition of set Γ_j of the sequence $\overline{\Gamma}$ and 1, for any Γ_{j-1} and $H, H \subset \Gamma_{j-1}$ we have

$$\pi(\alpha_n, H_n) < F(\Gamma_{j-1}), \forall \alpha_n \in W \setminus \Gamma_{j-1}.$$

Combining the two inequalities proves the lemma [5].

According to lemma 1 we can write:

$$\max_{\alpha_n \in W \setminus G} \pi(\alpha_n, H_n) \leq \max_{\alpha_n \in \Gamma_{p-1} \setminus G} \pi(\alpha_n, H_n), \tag{36}$$

where $G = \Gamma_p$ is the kernel of a *ominus*-monotonic system and Γ_{p-1} is the immediately preceding the kernel set in the sequence $\overline{\Gamma}$.

Lemma 3 *The weight of any element $k \in W$ that does not belong to $\Gamma_j, j = \overline{2,p}$, i.e. $k \in W \setminus \Gamma_j$ obtained by adding this element to this set is no larger that the value of the function $F(\Gamma)j)$. That is*

$$\pi(k, \Gamma_k \cup k) < F(\Gamma_j), \forall k \in W \setminus \Gamma_j, j = \overline{2,p}. \tag{37}$$

Proof: by contradiction. Assume that there exists $k \in W \setminus \Gamma_j$ such that

$$\pi(k, \Gamma_j \cup k) \geq F(\Gamma_j).$$

On the other hand, by monotonicity property of $\Gamma_j \cup k$:

$$\pi(i, \Gamma_j \cup k) \geq \pi(i, \Gamma_j) \geq F(\Gamma_j), \forall i \in \Gamma_j$$

Combining these two inequalities leads to

[5] Lemma 2 is true for defining sequence satisfying a general definition given in Section 2. The same holds for lemma 3.

$$F(\Gamma_j \cup k) = \min_{i \in \Gamma_j \cup} \pi(i, \Gamma_j \cup k) \geq F(\Gamma_j),$$

contradicting the statement (25) of theorem 6. The proof is complete.

According to lemma 2 we can now write

$$\pi(k, G \cup k) < F(G), \forall k \in W \backslash G. \tag{38}$$

Here we complete the discussion of properties of the kernel $G = \Gamma_p$ and other elements Γ_j of the sequence $\overline{\Gamma}$

Let us now move on to the description of the problems of structural analysis where, together with the kernel G, it is reasonable to use other sets of the sequence $\overline{\Gamma}$. Since the theory of monotonic systems and in particular the task of extracting an extremal subsystem are viewed here as a new approach to structural analysis, it is interesting to examine the possibility of solving the initial problem with some apriori constraints. Below we discuss two types of such constraints.

The first type of constraint arises when the problem of structural analysis specifies a desirable (necessary) size of the extremal subsystem or a desirable range for it.

The second type includes situations when the problem includes a set of elements that must be included into the subsystem.

In the practical problems of structural analysis both types of constraints may arise simultaneously. They are connected to apriori ideas, based on additional knowledge that might not be reflected in the data.

Consider a problem of extracting an extremal subsystem satisfying constraints of the first type.

Let $< W, \pi >$ be a $\ominus$-monotonic system. Fix number $n, n < N$. Consider a problem of finding a subset G_n of W such that the $F(H)$ attains a maximum there among all subsets of W that contain more than n elements. That is

$$F(G_n) = \max_H F(H), H \subseteq W, |H| > n. \tag{39}$$

The solution of this problem is given by the following theorem.

Theorem 11 *a) For any set Γ_j, $j = \overline{1,p}$ of sequence $\overline{\Gamma}$ and any set H, $H \subseteq W$ such that $|H| > |\Gamma_j|$ the following inequality holds:*

$$F(H) < F(\Gamma_j).$$

b) For any set Γ_j, $j = \overline{1,p}$ of sequence $\overline{\Gamma}$ and any set H, $H \subseteq W$ such that $|\Gamma_j| \geq |H| > |\Gamma_{j-1}|$ the following inequality holds

$$F(H) \leq F(\Gamma_j).$$

Let us now consider the problem of extracting an extremal subsystem of a monotonic system satisfying constraints of the second type. Again, let $< W, \pi >$ be a $\ominus$-monotonic system. Fix some set T of elements of W, $T \subseteq W$. Consider a problem of finding a subset G_T of W where $F(H)$ attains its maximum value among all subsets of W containing set T:

$$F(G_T) = \max_{T \subseteq H \subseteq W} F(H). \tag{40}$$

The elements of T can be called key elements. In other words, the problem is to extract an extremal subsystem of monotonic system containing the given key elements. The solution of this problem is given by the following theorem.

Theorem 12 *For any set $T, T \subseteq W$ and a set Γ_S of sequence $\overline{\Gamma}$ such that $T \subseteq \Gamma_S$ but $T \not\subseteq \Gamma_{S+1}$ the following inequalities hold:*

$$F(H) < F(\Gamma_S), \forall H \subseteq W, T \subseteq H, H \backslash \Gamma_S \neq \emptyset, \tag{41}$$

$$F(H) \leq F(\Gamma_S), \forall H \subseteq W, T \subseteq H, H \subseteq \Gamma_S. \tag{42}$$

The proof of theorems 11 and 12 follows directly from theorem 6, and therefore these theorems can be viewed as corollaries of theorem 6.

The properties of monotonic systems discussed here allow one to solve the problem of extracting an extremal subsystem of a monotonic system with additional constraints in the form of a set of elements that must be included in the subsystem or in the form of constraints on the size of the subsystem. In both cases, as follows from theorems 11 and 12 the natural solution is the smallest set Γ_j satisfying the specified constraints. Comparison between $F(\Gamma_j)$ and $F(G)$ demonstrates the degree of perturbation of the ideal solution by the introduction of additional constraints.

Note that in Section 2 the defining sequence of elements of monotonic system we introduced as some construct to be used for extracting the kernel, i.e. the maximum set $G, G \subseteq W$, where function $F(H)$, defined in (3) attains its global maximum. In this section we have demonstrated that the construction of the maximum defining sequence is used not only for finding the kernel, but for discovering the structure of monotonic system. Specifically, it allows us to extract a sequence of special subsets Γ_j, $j = \overline{1,p}$ of the initial set W that have properties similar to those of the kernel.

In light of this, it is natural, by analogy with definition of the last set $\Gamma_p = G$ as the kernel of monotonic system, to call other sets of sequence $\overline{\Gamma}$ quasi-kernels, and to call $\overline{\Gamma}$ itself - a sequence of quasi-kernels.

4 Algorithms for Extracting Kernels of Monotonic System

The existence of the defining sequence of elements of monotonic system, and therefore of the kernel, was assumed in all the definitions and and theorems introduced in the previous sections. To prove this it suffices to demonstrate a constructive procedure for computing such a sequence.

As was mentioned before, a monotonic sequence can have many distinct defining sequences. All of them yield the same kernel G. Also, all the maximal defining sequences generate the same sequence of quasi-kernels $\overline{\Gamma}$, according to the theorem 9.

We consider three different algorithms for extracting the kernel of a monotonic sequence. One of them constructs a general defining sequence, defined in Section 2. Another one constructs a maximal defining sequence, introduced in section 3. The third one computes a special kind of defining sequence.

Before we examine these algorithms, we discuss an iterative procedure used in both first and second algorithms [6].

Procedure $LAYER(\tau)$. The input of this procedure consists of an arbitrary set H together with the function $\pi(i, H)$ defined on this set and its elements, together with a scalar threshold τ. The pre-processing step includes computing all values $\pi(i, H), i \in H$ if there were not previously computed. Each iteration of this procedure consists of two steps. The first step consists of comparing $\pi(i, H)$ values of elements of H with the threshold τ and selecting those for which $\pi(i, H) \leq \tau$.

These elements are included in an arbitrary order into a sequence of elements of W that is being formed (while being removed from the set H).

In the second step, for each element i of $H', H' \subseteq H$ - a set of remaining elements - the value of $\pi(i, H')$ is computed instead of $\pi(i, H)$. After this the first step is repeated: elements of the set H' with function value $\pi(i, H') \leq \tau$ are selected, inserted into a sequence in an arbitrary order, and so on.

The last iteration of the procedure is determined by whether there are any elements of H with weight less than or equal to τ left. The result of the procedure is a set E, $E \subseteq H$, of the remaining elements with computed values $\pi(i, E) > \tau, \forall i \in E$. and an interval of sequence of elements of $W \backslash E$ in the form of several sequential groups, with elements in arbitrary order inside each group. In a special, but important, case procedure $LAYER(\tau)$

return an empty set, i.e. $E = \emptyset$, and all elements of H for intervals of the sequence.

Lemma 4 *Let $\tau_1 \geq \tau_2$, H_1 - result of $LAYER(\tau_1)$ on set W, H_2 - result of procedure $LAYER(\tau_2)$ on set W, H_3 - result of procedure $LAYER(\tau_1)$ on set H_2. Then $H_1 = H_3$.*

Proof: We first prove that $H_3 \subseteq H_1$. Because of monotonicity of $\pi(i, H)$ and since H_3 is the output of $LAYER(\tau_1)$, it follows that

$$\pi(i, W) \geq \pi(i, H_3) \geq \tau_1, \forall i \in H_3.$$

This means that after the first iteration of procedure $LAYER(\tau_1)$, the set $H' \supseteq H3$ and

$$\pi(i, H') \geq \pi(i, H_3) > u_1, \forall i \in H_3$$

Continuing in this fashion, we conclude that no element of H_3 can be thrown out at any step of $LAYER(\tau_1)$. Therefore $H_3 \subseteq H_1$.

Now we prove that $H_1 \subseteq H_3$. Clearly, $\pi(i, W) \geq \pi i, H_1 > \tau_1 \geq \tau_2, \forall i \in H_1$. That is, during the first iteration of $LAYER(\tau_2)$ on W no element of H_1 can be thrown out. By continuing this chain of reasoning analogously to above, we conclude that $H_1 \subseteq H_3$.

The two proved inequalities imply $H_1 = H_3$, QED.

Lemma 5 *Let $\tau_1 > \tau_2$, H_1 - result of $LAYER(\tau_1)$ on set H, H_2 - result of procedure $LAYER(\tau_2)$ on set H. Then, if $F(H_2) > \tau_2$ then $H_1 = H_2$. If $H_2 = \emptyset$, then $H_1 = emptyset$.*

Proof: It is easy to see that since $F(H_2) > \tau_1$, elements of H_2 could not have been thrown out by $LAYER(\tau_1)$. So $H_2 \subseteq H_1$. But since $F(H_1) > \tau_1$, $H_1 \subseteq H_2$. Therefore $H_1 = H_2$. The second statement of the lemma is proved analogously.

Algorithm $A1$ for constructing a general defining sequence (see Section 2). Assume that $\Gamma_1 = H = W$. Compute values $\pi(i, W)$ for all elements of W and determine the two values

$$\tau' = \min_{i \in W} \pi(i, W), \; \tau'' = \max_{i \in W} \pi(i, W).$$

Fix an arbitrary value τ so that $\tau' < \tau < tau''$, for example as the average of the two: $\tau = (\tau' + \tau'')/2$ The same procedure is used for recomputing the value of τ when values τ' and τ'' are changed during the execution of the algorithm.

Arbitrary j-th step. Run procedure $LAYER(\tau)$ on the set H (for the first step $H = W$). Two outcomes are possible.

1. The output set E is empty. That means that the threshold τ is too large. Then, set $\tau'' = \tau$, recompute τ and repeat the call to $LAYER(\tau)$ with the new value. At the same time, elements that were included into the sequence during this run of $LAYER$ are extracted from the sequence and returned to the current subset H. Thus we return to the situation with the two possible results.

2. Set E is not empty. In that case we fix the extracted elements in the sequence that is being formed and we declare E to be an element of sequence of sets $\overline{\Gamma}$, that is $\Gamma_{j+1} = E$. For this set we compute $F(\Gamma_{j+1})$. Note that $F(\Gamma_{j+1}) > \tau$. Once that is done we run $LAYER(F(\Gamma_{j+1}))$ on the set $H = \Gamma_{j+1}$.

As a result of the first situation, the set $H = \Gamma_{j+1}$ is the kernel Γ_p, i.e. $p = j + 1$. In the second case, when the resulting set $E \subseteq H$ is not empty, we move to the next $(j + 1)$st step with the set E. τ' is assigned the value of $F(H)$, τ is recomputed and $LAYER(\tau)$ is run on the last set E.

Theorem 13 *The set Γ_p obtained as the result of algorithm $A1$ is the defining set.*

Proof: We first prove that sets $\Gamma_j, j = \overline{1,p}$ produced by the algorithm form defining sets from a defining sequence. It suffices to show that each of these sets is a local maximum of the function $F(H)$ (see the Corollary 2). That is,

$$F(H) < F(\Gamma_j), \forall H \supset \Gamma_j, j = \overline{1,p}.$$

The set Γ_j is the result of applying procedure $LAYER(\tau)$, so $F(\Gamma_j) > \tau$. Assume $F(H) > F(\Gamma_j)$ for some $H \supset \Gamma_j$. Then $F(H) > \tau$, implying that

$$\pi(i, H) > \tau, \forall i \in H,$$

since

$$F(H) = \min_{i \in H} \pi(i, H).$$

Due to monotonicity of $\pi(i, H)$ we can state

$$\pi(i, W) \geq \pi(i, H) > \tau, \forall i \in H.$$

This means that during the execution of the first iteration of procedure $LAYER(\tau)$ none of the elements of H could have been removed. For the set of remaining elements $H', W \supseteq H' \supseteq H$ the following also holds

$$\pi(i, H') \geq \pi(i, H) > \tau, \forall i \in H.$$

Continuing this arguments leads to the conclusion that the result of $LAYER(\tau)$ can only be some set $H^0 \supseteq H$. But we are given that the result is some $\Gamma_j \subset H$. This is a contradiction. Therefore, it must be that

$$F(H) < \tau, \text{ i.e. } F(H) < F(\Gamma_j.$$

The proof of the fact that $\Gamma_R = G \not\subset \Gamma_p$ where G is the defining set follows from the definition of the kernel and from the description of algorithm $A1$ (the second case).

Algorithm $A2$ for constructing maximal defining sequence (see section 3) consists of $p \leq N = |W|$ steps (parameter p is not foxed however but is automatically determined in the process of execution of the algorithm). Each step consists of two stages.

In the first stage, the set of elements remaining before step j is declared to be a set Γ_{j+1}, in particular $\Gamma_1 = W$. The value $F(\Gamma_j) = \min_{i \in \Gamma_j} \pi(i, \Gamma_j)$ computed on the previous step (prior to the first step all the values of $\pi(i, W)$, and also $F(W)$ are computed) is taken as the j-th value of threshold ϵ_j. Thus, for all steps except the first one, the first stage is purely declarative and does not involve any computations.

Stage 2 consist of applying procedure $LAYER(F(\Gamma_j))$ to the set Γ_j. Two results are possible: 1. The set E is empty. In that case Γ_j is the kernel, i.e. $p = j$, and the algorithm is done. 2. The set E is not empty. In that case we start $(j + 1)$-st step of the algorithm with the set E and the threshold $F(E) = \min_{i \in E} \pi(i, E)$.

Theorem 14 *1. The result of algorithm* $A2$ *is the maximal defining sequence. 2. The sequence* $\overline{\Gamma^1}$ *obtained as a result of executing algorithm* $A1$ *is a subsequence of sequence* $\overline{\Gamma^2}$ *obtained as a result of executing algorithm* $A2$.

Proof: 1. The fact that the constructed sequence is a defining sequence is proved analogously to the proof in theorem 13 for sequence $\overline{\Gamma^1}$. It remains to show that the condition (24) holds. Element $\alpha_k \in {\Gamma_j}^2 \backslash {\Gamma_{j+1}}^2$ is extracted during some iteration of $LAYER(F({\Gamma_j}^2))$. Because of monotonicity of $\pi(i, H)$, and since the set H_k is a subset of θ_l, a result of l-th iteration of $LAYER(F({\Gamma_j}^2))$ during which the element α_k is extracted, it follows that

$$F({\Gamma_j}^2) > \pi(\alpha_k, \theta_l) \geq \pi(\alpha_k, H_k).$$

Since this holds for an arbitrary choice of set ${\Gamma_j}^2$ and the element α_k, the proof is complete.

2. Based on lemma 4 the results of $LAYER(\tau)$ on W and on any set Γ_j are the same. Therefore the set ${\Gamma_k}^2$ is the result of applying procedure $LAYER(\tau)$ to a set W for some τ. If $\tau = F({\Gamma_j}^2)$, then ${\Gamma_k}^1 = {\Gamma_{j+1}}^2$. If $\tau \neq F({\Gamma_j}^2)$ for any j, then there exist sets ${\Gamma_j}^2$ and ${\Gamma_{j+1}}^2$ such that

$$F({\Gamma_j}^2) < \tau, F({\Gamma_{j+1}}^2) > \tau \tag{43}$$

Sequential application of lemmas in this case also leads to

$${\Gamma_k}^1 = {\Gamma_{j+1}}^2.$$

Therefore any set of the sequence $\overline{\Gamma^1}$ is a set from the sequence $\overline{\Gamma^2}$, QED.

Below we describe an algorithm for extracting a so called strict defining sequence. While its execution requires a larger number of operations that for algorithms $A1$ or $A2$, its description is much simpler.

Algorithm $A3$ constructs a sequence $I =< i_1, .., i_M >$ of elements of W such that for any $k, 1 \leq k \leq N$ if $H_k = i_k, \ldots, i_N$ then the following holds:

$$\pi(i_k, H_k) = \min_{i \in H_k} \pi(i, H_k). \tag{44}$$

In the situation where the minimum in (44) is attained at several elements at the same time, any one of them is chosen. The desired set G is chosen to be the largest set H_m for which

$$\pi(i_m, H_m) \geq \pi(i_k, H_k), \forall k = \overline{1, N} \tag{45}$$

For $\oplus$-monotonic system minimum is substituted for maximum in (44) and "$\leq$" is replaced by "$\geq$" in (45).

In other words, constructing a defining sequence is accomplished via sequential removal of an element from a set of remaining elements, i.e. elements

i_k with minimal value of $\pi(i_k, H_k)$ is removed from H_k on k-th step. All the weights are recalculated after each step. The elements in the order of removal constitute the desired sequence I and a corresponding sequence of nested sets $\overline{H} =< H_1, \dots, H_N >$, where

$$H_1 = W, H_2 = H_1 \backslash i_1, \dots, H_N = i_N. \tag{46}$$

The algorithms completes when all elements of set W are arranged in sequence I. At the same time, the extremal value of $\pi(i_m, H_m)$ of the function $\pi(i_k, H_k)$ together with the corresponding element $g = I_m$ and the set $G = H_m$ is recorder:

$$\pi(i_m, H_m) = F(G) = \max_{k=\overline{1,N}} \pi(i_k, H_k). \tag{47}$$

The fact that G is chosen as the largest set satisfying (47), i.e. the set with the maximal value of f=the function $F(H)$ that occurs first n the sequence $\overline{H}$ can be restated as follows [6]:

$$F(H_k) < F(G), \forall H_k \supset G, \forall k = \overline{1, m-1} \tag{48}$$

$$F(H_k) \leq F(G), \forall H_k \subseteq G, \forall k = \overline{m, N} \tag{49}$$

Let us consider in more detail how the construction of the sequence of quasi-kernels $\overline{\Gamma} =< \Gamma_j, j = \overline{1,p} >$ is accomplished. The construction of sequence I of elements of W implies concurrent construction of the sequence of nested sets $\overline{H}$ and of the scalar sequence of thresholds $\delta =< \delta_1, \dots, \delta_N >$, where

$$\delta_1 = \pi(i_1, W) = F(H)\delta_k = \max[\delta_{k-1}, \pi(i_k, H_k) = F(H_k)]. \tag{50}$$

The expression (50) allows us to highlight the points of sequence $\overline{H}$ that correspond to a change (in this case to an increase) of the threshold value. If we denote then by Γ_j,$j = \overline{1,p}$,$(p \leq N = |W|)$ we obtain a sequence of nested sets $\overline{\Gamma}$ that is a subsequence of a subsequence of sets H for which conditions (5) and (24). Theorem 15 shows that this is exactly the sequence $\overline{\Gamma}$ whose existence is assumed in the definition of maximal defining sequence.

Theorem 15 *The sequence I constructed by algorithm $A3$ is a maximal defining sequence of a ominus-monotonic system $< W, \pi >$.*

[6]For a $\oplus$-monotonic system instead of (48) and (49) the following inequalities hold: $F(H_k) > F(G), \forall H_k \supset G, \forall k = \overline{1, m-1}$ $F(H_k) \geq F(G), \forall H_k \subseteq G, \forall k = \overline{m, N}$

Proof: We need to prove that sets $\Gamma_j, j = \overline{1,p}$ of the sequence $\overline{\Gamma}$ obtained in the process of constructing the sequence I satisfy properties (5), (6) and (24) according to the definition of the maximal defining sequence.

The fact that properties (5) and (24) hold is clear since sets Γ_j are constructed by the above algorithm when the threshold value δ_k is increased (see (50)). Let us prove that (6) holds. As a result of executing algorithm $A3$ for constructing sequence I we have:

$$F(H_k) \leq F(G), \forall H_k \subseteq G, (\forall k = \overline{m, N}).$$

We need to prove that

$$F(H) \leq F(G), \forall H \subseteq G,$$

i.e. that this inequality is true for any subsets of G, not just for those in the sequence $\overline{H}$.

Assume that the opposite is true. Let $H \subset G$ be such that

$$F(H) > F(G) \tag{51}$$

Let H_n be the smallest set of sequence $\overline{H}$ containing H, i.e. $H_n \supseteq H$, $H_{n+1} \not\supseteq H$. Clearly, $i_n \in H, i_n \in H_n, i_n \in H_{n+1}$. Then, considering the definition of F, monotonicity property and relation (44), we can establish the following:

$$F(H) \leq \pi(i_n, H) \leq \pi(i_n, H_n) \leq F(G)$$

But this contradicts (51). The proof is complete.

The described algorithm allows construction of a strict defining sequence. Its main properties are the satisfaction of (44) and the extraction of the kernel that satisfies (48) and (49). The application of this algorithm requires some clarification.

Definition. Elements i_k and i_n, $k > n$ of the defining sequence I of a $\ominus$-monotonic system are called I-equal if

$$\pi(i)k, H_k) = \pi(i_n, H_n) = \min_{i \in H_n} \pi(i, H_n) = F(H) \tag{52}$$

(Accordingly, for a $\oplus$-monotonic system min is substituted for max).

The algorithm $A3$ constructs a defining sequence exactly up to I-equal elements. It cannot be known in advance how different the two sequence, resulting from the choice of i_k or i_n from some group of I-equal elements, will be. It is known however (Theorem 9) that the sequence of quasi-kernels of two different defining sequences coincide and therefore are independent of a choice of I-equal elements. This result is confirmed by the following lemma.

Lemma 6 *If i_k and i_n, $k > n$ are I-equal elements of sequence I of $\ominus$-monotonic system $< W, \pi >$ then there can be no set Γ_S in the sequence $\overline{\Gamma}$ such that $i_k \in \Gamma_S$ but $i_n \notin \Gamma_S$. In other words, there exists a set $\Gamma_j \in \overline{\Gamma}$ such that $i_k, i_n \in \Gamma_j \backslash \Gamma_{j+1}$ or $i_k, i_n \in \Gamma_p$.*

Proof: Assume the opposite. Let $\Gamma_S \in \overline{\Gamma}$ such that $i_k \in \Gamma_S$ but $i_n \notin \Gamma_S$. Then, clearly $\Gamma_S \subset H_n$. Using the theorem 6 on set H_n, $H_n \backslash \Gamma_S = \emptyset$ we have the following sequence of inequalities

$$\pi(i_n, H_n) < F(\Gamma_S) \leq \pi(i_k, \Gamma_S) \leq \pi(i_k, H_n).$$

At the same time, according to the definition of I-equal elements, we have

$$\pi(i_k, H_n) = \pi(i_n, H_n)$$

We have a contradiction. The proof is complete.

In other words, the choice of a particular element among several I-equal ones does not affect the construction of the sequence $\overline{\Gamma}$ of quasi-kernels and on the extraction of the kernel G. At the same time, for the proofs of theorems about the properties of monotonic systems and for practical applications it is necessary to choose an arbitrary but fixed method of constructing a defining sequence. We fix the following rule. *Rule 1.* If, in the process of constructing a defining sequence of $\ominus$-monotonic system according to the algorithm described above, at the n-th step we encounter I-equal elements, we choose an element with the smallest input number, while for $\oplus$-monotonic system we choose an element with the largest input number. (It is assumed that the elements of set W are represented in a for of some list and can thus be ordered).

The sequence I constructed according to the above algorithm, using this rule, is called fixed.

References

[1] Algorithmic and programming methods of applied statistical analysis. M., 1980

[2] S. M. Borodkin. Optimal clustering of interconnected ordered objects. - Avtomatika i telemekhanika, 1980, Vol. 2, pages 165-172.

[3] E.M. Braverman, N. E. Kiseleva, I. B. Muchnik, S. G. Novikov. Linguistic approach to the problem of processing large datasets. - Avtomatika i telemekhanika, 1974, Vol. 11, pages 73-88.

[4] E. N. Kuznecov, I. B. Muchnik. Monotonic systems for analysis of organizational structures. - In "Methods of analysis of multidimensional economic information". Novosibirsk, 1981. pages 71-84

[5] B. G. Mirkin. Analysis of qualitative properties and structures. M. 1980.

[6] J. E. Mullat. Extremal subsystems of monotonic systems. I, II, III. - Avtomatika i telemekhanika, 1976, Vol. 5 pages 130-139; 1976, Vol. 8, pages 169-178; 1979 Vol. 1, pages 65-83

[7] I. B. Muchnik, A. A. Oslon. Construction of an approaximating factor for the connectivity matrix. - Avtomatika i telemekhanika, 1980, Vol. 4 pages 89-96;

[8] Accounting for uncertainty factors in industrial planning, Novosibirsk, 1979.

[9] J. Mullat, L. Vyhandu. Monotonic systems in scene analysis. - In: Symposium "Mathematical processing of cartographic data". Tallinn, 1979, pages 63-66.