MODELING OF BEHAVIOR AND INTELLIGENCE *

Maximization of generalized characteristics of functions of monotone system

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We consider monotone system in which the value of the extremand criterion is determined by the worst element in some part of a given subset. Examples of practical data aggregation problems leading to such system are given. It is shown that the so-called convex geometries (antimatroids) constitute the accessible family of subsets that are responsible for the effectiveness of the extermization algorithms of such criteria.

1. INTRODUCTION

Various aggregation problems for finite a finite set of empirical elements (some problems of cluster analysis, simultaneous aggregation of objects and attributes using the data matrix, identification of analogs, etc) may be formulated as problems of extremizing the characteristic functions of monotone systems [1]. This formalization is convenient because it leads to simple polynomial-time algorithm solving these applied problems by a uniform procedure.

The construction of these algorithms relies on quasiconvexity of the characteristic functions of monotone systems. Yet it remains unclear what properties of monotone systems are in fact responsible for effective extremization of quasiconvex functions. It is desirable to identify these properties and utilize them to construct a generalization of monotone systems.

The well-known result of Rado and Edmons regarding extremization of linear function on an independence system provides an example of the description of systems whose properties guarantee effective extremization of characteristic functions. A systematic comparison of this example with the monotone system construction has established an analogy between matroids and antimatroids. This analogy has produced the sought generalization of monotone systems and a Rado-Edmons type description of the problem of extremization of quasiconvex functions of such systems.

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2. STATEMENT OF THE PROBLEM

To construct a monotone system on a finite set W, we define a parametric family of functions $\pi(x,H)$ (H is a parameter $H \subseteq W$, $x \in H$) such that all $x \in H' \subseteq H \subseteq W$ we have

$$\pi(x,H') \leq \pi(x,H). \tag{1}$$

We then use this family to construct a characteristic estimate $F(H)$ of an arbitrary set $H \subseteq W$ in the form

$$F(H) = \min_{x \in H} \pi(x,H). \tag{2}$$

The problem is to maximize the characteristic function (2). This problem is adapted to various applications by varying the description of the set W and the construction of the function $\pi(x,H)$ given in this description. In all these cases, however, the estimate of the subset H is constructed as the estimate of the "worst" element by (2).

Yet in some applications the estimate of H based on the worst element in this subset does not correspond to the physical content of the problem. It is required to estimate the subset using the elements from some part of the subset, i.e., the characteristic function should be defined in the form

$$F_{\varphi}(H) = \min_{x \in \varphi(H)} \pi(x,H), \tag{3}$$

where $\varphi : 2^W \to 2^W$ is the so-called choice operator [2] satisfying the condition $\varphi(H) \subseteq H$.

A function of the form (3) will be called the generalized (φ-generated) characteristic function of the monotone system $\langle W, \pi, F \rangle$.

Consider an example of a data analysis problem, which corresponds to this definition of characteristic function.

Example. W is a set of points in the Euclidean space $\Re^n$ and $\varphi(H)$ is the set of extreme points of the set H (the convex hull of H). Define $\pi(x,H) = \sum_{y \in H} r_{xy}$ where r is the Euclidean distance. Then the maximization problem (3) involves finding H^* in which all extreme points are maximally distant. Substantively, H^* generates the sharpest partition (compared to other sets) into a "core" $\left(H^* - \varphi(H)\right)$ and a "shell" $\left(\varphi(H^*)\right)$.

The introduction of generalized characteristic functions naturally leads to examination special families of subsets – convex geometries (antimatroids) [3,4]. The generation of antimatroids is described in Sec.2, where we also prove that it is only of these systems that generalized characteristic functions have the quasiconvexity property, which is important for the construction of effective extremization algorithms.

In Sec. 3 we construct a maximization algorithm for the functions (3). Comparison of this algorithm with greedy procedure of extremization of the sum-of-weights function establishes an analogy between the role of antimatroids among all accessible [4] systems and the role of matroids among all independence systems [5,6]. We prove the duality theorem for the analog of the Lagrange function for the extremization of functions (3) on convex geometries. We conclude Sec. 3 with an examination of the problem of isolating the best (in this sense) submodel of an econometric model identified from statistical data.

The theorems are proved in the Appendix.

3. CONVEX GEOMETRIES AS DOMAINS OF DEFINITION OF GENERALIZED CHARACTERISTIC FUNCTION

Let $\varphi(H)$ be a choice operator. We use it to construct a family B of subsets H ($H \subseteq W$) by the following algorithm.

Generation Algorithm (GA)

Step 1. $W \in \mathrm{B}$.

Step i. Enumerate all the sets H whose membership in B was established in step $(i-1)$; each of these subsets generates $|\varphi(H)|$ subsets of B by the condition $x \in \varphi(H) \Leftrightarrow (H - x) \in \mathrm{B}$; if step i does not add new nonempty subsets to B, then stop; otherwise to step $i+1$.

Definition 1. The system (W, B) constructed by GA is called φ-generated.

It is easy to see that a φ-generated system has the following properties, which are essential for our analysis:1

[1] In what follows, $\{\ \}$ are omitted from the notation of single-element sets $\{x\}$.

1) $W \in \mathrm{B}$;

2) $\forall H \in \mathrm{B}$, $H \neq \varnothing$ we have $(H - x) \in \mathrm{B} \Leftrightarrow x \in \varphi(H)$;

3) $\forall H \in \mathrm{B}$, $H \neq \varnothing$ there is $x \in W - H$ such that $H \cup x \in \mathrm{B}$.

For any set system (W, B) having property 3), an operator φ can be constructed by rule 2). Moreover, property 1) also follows from 3). Applying these considerations to GA, we obtain

Proposition 1. The set system (W, B) is φ-generated if and only if it satisfies property 3).

Definition 2. The system $(W, \overline{\mathrm{B}})$ is called complementary to (W, B) if $\overline{\mathrm{B}} = \{H \mid W - H \in \mathrm{B}\}$.

Definition 3 [4]. The set (W, F) is called accessible if:

1) $\varnothing \in \mathrm{F}$;

2) $\forall H \in \mathrm{F}$, $H \neq \varnothing$ there exists $x \in H$, such that $H - x \in \mathrm{F}$.

Note that in this definition property 1) follows from 2). Comparison of property 2) of Definition 3 with property 3) of Definition 1 leads to

Proposition 2. The system (W, B) is φ-generated if and only if the complementary system $(W, \overline{\mathrm{B}})$ is accessible.

Definition 4 [2]. The operator φ is called hereditary on the set system (W, P) if $\forall H'$, $H \in \mathrm{P}$ we have

$$(H' \subseteq H) \Rightarrow H' \cap \varphi(H) \subseteq \varphi(H'). \tag{4}$$

It is easy to check that all operators $\varphi(H)$ used in our examples (see Introduction and the end of Sec. 3) are hereditary.

Definition 5 [4]. An accessible set system closed relative to union is called an antimatroid; the complementary system of an antimatroid is called a convex geometry.

THEOREM 1. The set system (W,B) is φ-generated by some hereditary choice operator φ if and only if it is a convex geometry.

From the definition of hereditary operator φ and the fact that the system generated by a hereditary operator φ is a lower semilattice 2 we easily obtain

Proposition 3. Let (W,B) be the system generated by the hereditary operator φ. Then $\forall A,B \in \mathrm{B}$, such that $|B| < |A|$ we have $\varphi(A) \cap (A-B) \neq \varnothing$.

Remark. Antimatroids are a particular case of greedoids, which have been studied by many authors, e.g., in [4, 7-10]. A greedoid is an accessible system (W,F) with the exchange property:

$\forall A,B \in \mathrm{F}$, such that $|A| < |B|$ there exists $x \in B - A$ such that $(A \cup x) \in \mathrm{F}$.

It is easy to show that a set system is a greedoid if and only if its complementary system is φ-generated by the operator φ that satisfies property (a).

Definition 6. The family $\mathrm{M} = \{H | H \in \mathrm{B}, \varphi(H) = \varnothing\}$ is called the set of dead-end vertices of the system (W,B), and the system $(W,\mathrm{B}-\mathrm{M})$ is called a φ-generated system without dead-end vertices.

On the system $(W,\mathrm{B}-\mathrm{M})$ we define the function

$$F_\varphi(H) = \min_{x \in \varphi(H)} \pi(x,H),\; H \in \mathrm{B}-\mathrm{M}, \tag{5}$$

which is called φ-generated characteristic function. 3

Definition 7. The function $P(H)$ defines on a φ-generated set system without dead-end vertices $(W,\mathrm{B}-\mathrm{M})$ is called quasiconcave if for all triples of sets $(H_1,H_2,H_1 \cup H_2)$ such that all their elements are contained in the family $(\mathrm{B}-\mathrm{M})$ we have $P(H_1 \cup H_2) \geq min(P(H_1),P(H_2))$.

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[2] The fact that the family B is a lower semilattice 2^W is established in the proof of Theorem q in the Appendix.

[3] The function (5) is obviously a restriction to the family $(\mathrm{B}-\mathrm{M})$ of the generalised characteristic function (3) defined on the entire Boolean 2^W if $\pi(x,H)$ satisfies condition (1); in what follows, we consider only such functions $F_\varphi(H)$.

We know [11,12] that if $\varphi(H) = H$ and $\pi(x,H)$ satisfies condition (1), then $F_\varphi(H)$ is quasiconcave on the entire Boolean 2^W; in [13] this property of $F_\varphi(H)$ with $\varphi(H) = H$ is essentially used in the construction of maximization algorithms on the entire 2^W and on arbitrary upper subsemilattices of 2^W.

Fix a choice operator φ. Assume that the system $(W, \mathrm{B} - \mathrm{M})$ generated by this φ contains only one triple $(H_1, H_2, H_1 \cup H_2)$ such that

$$H_1 \neq \emptyset,\ H_2 \neq \emptyset \text{ and } H_1, H_2, H_1 \cup H_2 \in \mathrm{B} - \mathrm{M}. \tag{6}$$

Then we have the following theorem.

THEOREM 2. For the φ-generated characteristic function of any monotone system [i.e., a monotone system with an arbitrary function (1)] to be quasiconcave on $(W, \mathrm{B} - \mathrm{M})$, it is necessary and sufficient that φ is a hereditary operator on (W, B).

We have thus established a correspondence between the property of quasiconcavity for the functions (5) and the property of being a convex geometry for φ-generated set systems. This correspondence is essentially used in the next section, which considers extremization of the functions (5).

4. MAXIMIZATION ALGORITHMS FOR GENERALIZED CHARACTERISTIC FUNCTIONS AND PROPERTIES OF THEIR EXTREMAL SUBSETS

Consider a procedure (we call it F-procedure) which, given the φ-generated function F_φ of the monotone system $\langle W, \pi, F_\varphi \rangle$, isolates a unique element H^* on the φ-generated system $(W, \mathrm{B} - \mathrm{M})$.

F-Procedure.

0. $i \leftarrow 1$, $H_i \leftarrow W$.
1. Choose an arbitrary $x_i \in \varphi(H_i)$ which satisfies the system inequalities $\pi(x_i, H_i) \leq \pi(y, H_i)$ for all $y \in \varphi(H_i)$. If $\varphi(H_i) = \emptyset$, then $m \leftarrow (i-1)$ and go to 4.
2. $H_{i+1} \leftarrow H_i - x_i$, $i = i + 1$.
3. Go to 1.
4. Find $H^* = arg \max\limits_{i=1,m} \pi(x_i, H_i)$ that corresponds to the minimum number in the series $\langle 1,2,\ldots,m \rangle$ on which this maximum is attained.

This algorithm in a certain sense is an antigreedy procedure: instead of adding the best element y to an existing set (greedy), it removes the worst element from the existing set. 4

4 Such algorithms are called “worst off greedy” in [9].

THEOREM 3. The result H^* produced by the F-procedure is a solution of the maximization problem of the φ-generated characteristic function $F_\varphi(H)$ for any monotone system $\langle W,\pi,F_\varphi\rangle$ on the φ-generated system $(W,\mathrm{B}-\mathrm{M})$ if and only if the choice operator φ is hereditary or, in other words, the system $(W,\mathrm{B}-\mathrm{M})$ is a convex geometry.

Theorem 3 is an analog of the well-known result of Rado and Edmonds [5,6] on the relationship of the greedy algorithm to a set system called matroid. In order to trace directly this analogy, we constructed Table 1 in which the left column lists the main elements from [5,6] and the right column lists their analogs in our problem.

Note that the F-procedure (the right-hand column in Table 1) differs from greedy algorithm (the left-hand column) in that, first, it does not check in step 1 for nonnegativity of the function value on the discarded element. Another essential difference is the presence of step 4 in the F-procedure, i.e., contrary to the greedy algorithm, which constructs only a segment of the maximum chain whose leading set is the solution, the F-procedure constructs the entire maximum chain and only then selects a solution from this chain.

Goecke's modified greedy procedure [10] for the maximization of a linear objective function differs from the standard greedy procedure in these two respects.

Goecke's Procedure.

0. $i \leftarrow 1,\ I_i \leftarrow \varnothing$.
1. Choose $x_i \in W - I_i$ such that $I_i \cup x_i \in \mathrm{F}$ and $\omega(x_i) \geq \omega(y)$ for all $y \in W - I_i$; if no such x_i exists, then set $m \leftarrow (i-1)$ and go to 4.
2. $I_{i+1} \leftarrow I_i \cup x_i,\ i \leftarrow i+1$.
3. Go to 1.
4. Find $I^* = arg \max\limits_{i=1,m} \omega(I_i)$, corresponding to the minimum i.

A generalization of the Rado-Edmods theorem for this procedure is proved in [10]. 5

[5] Thus, there is a complete analogy between our results and the results of [10]. Matroids, however, are better known object, and for this reason Table 1 presents the analogy with the results [5,6].

TABLE 1

Matroids	Convex geometries
1) Independence system (W,F): $\varnothing \in \mathrm{F}$ and for each $I \in \mathrm{F}$ and every $x \in I$, $I - x \in \mathrm{F}$.	1) φ -generated system
2) Matroid: a) independence system, b) exchange property: if $A, B \in \mathrm{F}$ and $\lvert A\rvert < \lvert B\rvert$ then there exists $x \in B - A$ such that $A \cup x \in \mathrm{F}$	2) Convex geometry: a) φ -generated system b) φ a hereditary operator
3) Linear set objective function: $\omega(I) = \sum_{x \in I} \omega(x)$, $I \in \mathrm{F}$	3) Quasiconcave generalized characteristic function of monotone system (5)
4) Greedy algorithm: 0. $I \leftarrow \varnothing$ 1. Choose $x \in W - I$ such that $I \cup x \in \mathrm{F}$, $\omega(x) \geq 0$; $\omega(x) \geq \omega(y)$ for all $y \in W - I$ such that $I \cup y \in \mathrm{F}$; if such x does not exists, then $I_W \leftarrow I$ and STOP. 2. $I \leftarrow I \cup x$. 3. Got to 1. I_W is the solution.	4) F -procedure (antigreedy algorithm)
5) Rado-Edmonds theorem: The result I_W produced by greedy algorithm is the solution of the maximization problem for any linear objective function on the independence system (W,F) if and only if this system is a matroid.	5) Theorem 3

Greedoids play the role of independence systems and a special class of Gaussian elimination greedoids plays the role of matroids. These Gaussian elimination greedoids are defined as follows:

a) F is a greedoid;

b) for any $B \in \mathrm{F}$, such that $z \in B$, $x, y \notin B$, it follows from $(B - z) \cup \{x, y\} \in \mathrm{F}$ that either $(B - z) \cup y \in \mathrm{F}$, or $B \cup y \in \mathrm{F}$.

It is easy to see that if the hereditary operator is defined on the entire Boolean 2^W and it additionally satisfies the nonempty choice property

$$\varphi(H) = \varnothing \Leftrightarrow H = \varnothing \tag{7}$$

then the set system (W,B) generated by this operator is the union of some collection of maximum chains on 2^W.

Proposition 4. If the hereditary operator φ satisfies condition (7) on the Boolean 2^W, then the F-procedure maximizes the function F_φ (5) on the set system $(W, 2^W - \varnothing)$.

This proposition easily follows from the maximality of the chains and from the fact that in each step the F-procedure removes an element x_k from the set H_k which is "unpromising" on the entire interval $[x_k, H_k]$. This proposition is obviously essential for algorithmic solution of the problem considered in our examples (see Introduction and the end of Sec. 3).

The sequence of sets $(H_1, H_2, ..., H_m)$ and elements $(x_1, x_2, ..., x_m)$ from F-procedure will be called defining sequence, by analogy with [1], where such constructs are used in the algorithm extremizing the characteristic function F (2) for ordinary monotone systems.

The next theorem characterizes the solutions of the maximization problem for the function (5).

THEOREM 4. Let (W, B) be a φ-generated system and φ a hereditary operator on this system. Then we have the following assertions:

1) there exists a unique maximum-cardinality solution $\widehat{H}$ of the maximization problem for F_φ (5) on $(W, \mathrm{B} - \mathrm{M})$;
2) $\widehat{H}$ belongs to any defining sequence constructed by the F-procedure;
3) any solution of the maximization problem for F_φ (5) on $(W, \mathrm{B} - \mathrm{M})$ is a subset of the set $\widehat{H}$;
4) if H_1 and H_2 are solutions of the maximization problem F_φ (5) on $(W, \mathrm{B} - \mathrm{M})$ and $H_1 \cup H_2 \in \mathrm{B}$, then $H_1 \cup H_2$ is also a solution of this problem.

Denote by Ω the set of all sequences of the elements of the sets $W - H_0$ that are discarded by the GA during the construction of (W, B). Here H_0 is the zero of sublattice B Define the function of two variables $\Pi : \Omega \times (\mathrm{B} - \mathrm{M}) \rightarrow \Re$ using the function $\pi(x, H)$ that satisfies (1):

$$\Pi(J, H) = \pi(x_{J,H}, H),$$

where $x_{J,H}$ is the first element of the set $\varphi(H)$ that occurs in the sequence $J \in \Omega$. The following theorem shows that the function Π acts as the Lagrange function 6 for the function $F_\varphi(H)$.

[6] The results of this section are a generalization and refinement of the results of [1,4]. In the construction of the duality theorem in [14] it is stated that the scheduling theorems proved in [15] are corollaries of the assertion that the F-procedure and the duality theorem hold for an arbitrary φ-system. Our paper shows that this assertion is not true. Yet the results of [15] follow directly from Theorems 3 and 5 of this section.

THEOREM 5. The subset H^* is a solution of the maximization problem for $F_\varphi(H)$ on the system $(W, \mathrm{B}-\mathrm{M})$, if and only if there exists a sequence $J^* \in \Omega$ such that for all $J \in \Omega$, $H \in \mathrm{B}-\mathrm{M}$ we have

$$\Pi(J, H^*) \geq \Pi(J^*, H^*) \geq \Pi(J^*, H). \tag{8}$$

COROLLARY._The defining sequence $(x_1, x_2, ..., x_m)$ constructed by the F-procedure is the left segment of the sequence J^* that satisfies (8). We also have the equality

$$F_\varphi(H^*) = \Pi(J^*, H^*).$$

In conclusion, let us illustrate the application of the proposed construction in economic modeling [16]. Assume that the elements of the set W are the nodes of some economic system and the interaction between these nodes are described by the loopless directed graph (W, G) where G is the set of arcs. The system (W, G) is connected with the external environment (other systems) in the following way. External actions may be applied to each node, but there is a distinguished set of outputs $\varphi(W) \subseteq W$ which may communicate with the environment, i.e., deliver the products of the system's activity to the outside world.

A subsystem of the system (W, G) is any subgraph (H, G), where $H \subseteq W$. The external environment of the subsystem (H, G) is the external environment of the system (W, G) plus the elements of the subset $W - H$. An output of the subsystem (H, G) is therefore an element y of the set H such that either $y \in \varphi(W)$ or an arc leads from y to $W - H$. The set of outputs of the subsystem (H, G) will be denoted by (H).

The operator φ defined in this way is clearly a hereditary choice operator. Also assume that it satisfies property (7). This is a natural property, e.g., for production systems: any subsystem communicates with its external environment.

We say that the node $x \in H$ influences the node $y \in H$ if there is a path from x to y in the subgraph (H,G). Moreover, assume that each node is characterized by some scalar variable (e.g., its yield) and that a number of observations of the node characteristics have been made, each observation comprising measurements of all characteristics made at some time.

Given this information, it is required to identify a subsystem of nodes such that the characteristic of each of its outputs can be predicted, in liner regression sense, from the characteristics of the influencing nodes in the subsystem and the corresponding predictor is the best (in some sense) compared to all other subsystems.

This problem can be formulated as follows. Assume that for any $H \subseteq W$ and any $x \in H$,

$$\pi_\rho(x,H) = \begin{cases} \rho(x,H_x), & \text{if} \quad H_x \neq \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

where H_x is the set of elements influencing x, $\rho(x,H_x)$ is the set correlation coefficient between x and H_x.

We easily see that $\langle W,\pi_\rho \rangle$ is a monotone system. We define is φ-generated characteristic function

$$F_\rho(H) = \min_{x \in \varphi(H)} \pi_\rho(x,H).$$

The econometric problem formulated above can be formalized as the problem of finding the subset H^* that maximizes the function $F_\rho(H)$.

Since $\varphi(H)$ satisfies property (7) (see above), Proposition 4 indicates that this problem is solved by the F-algorithm.

APPENDIX

Proof of Theorem 1.

Let $(W,\mathcal{B})$ be a set system φ-generated by the hereditary operator φ. We will show that the complementary system $(W,\overline{\mathcal{B}})$ is an antimatroid. Proposition 2 implies that $(W,\overline{\mathcal{B}})$ is an accessible system. Therefore, it remains to show that $(W,\overline{\mathcal{B}})$ is closed under union or, equivalently, that $(W,\mathcal{B})$ is closed under intersection, i.e., is a lower subsemilattice of the Boolean.

Let $A,B\in\mathcal{B}$, $A\neq\varnothing$, $B\neq\varnothing$. By GA, there exists a sequence $(x_1,x_2,...,x_k)$ of distinct elements in W whose successive removal produces the set A $(W-\{x_1,x_2,...,x_k\}=A)$.

Take the first element x_i in $(x_1,x_2,...,x_k)$ such that $x_i\in B$. If no such element exists, then $B\subseteq A$ and $B=A\cap B$. Otherwise, $B\subseteq W-\{x_1,x_2,...,x_k\}$. Using the hereditary property of the operator φ, we obtain that $x_i\in\varphi(B)$.

Let $U=B-x_i$. Take the first element x_j of the sequence $(x_{i+1},x_{i+2},...,x_k)$ such that $x_j\in B$. If no such element exists, then $B-x_i\subseteq A$ and $U=A\cap B$. Otherwise, as before $x_j\in\varphi(B-x_i)$.

Repeating this procedure, we successively remove from B all the elements of the set $A-B\cap A$. In each step we obtain $U\in\mathcal{B}$, i.e., in the last step $U=B\cap A\in\mathcal{B}$.

We will now show that for any antimatroid $(W,\mathcal{B})$ the corresponding system $(W,\mathcal{B})$ is φ-generated by some hereditary operator φ. Proposition 2 shows that $(W,\mathcal{B})$ is a φ-generated system. We know [4] that antimatroids have the interval property without upper bounds: $\forall A,B\in\mathcal{B}$ such that $B\subseteq A$, $\forall x\in W-A$ we have $B\cup x\in\overline{\mathcal{B}}\Rightarrow A\cup x\in\overline{\mathcal{B}}$. By Proposition 2 and Definition 4 of the operator φ for the system $(W,\mathcal{B})$, this means that φ is a hereditary operator on $(W,\mathcal{B})$. ■

Proof of Theorem 2.

Sufficiency. Consider a triple that satisfies (6). Let

$$i_A = arg \min_{i \in \varphi(A)} \pi(i,A);$$

$$i_B = arg \min_{i \in \varphi(B)} \pi(i,B);$$

$$i_{A \cup B} = arg \min_{i \in \varphi(A \cup B)} \pi(i,A \cup B).$$

For definiteness let $i_{A \cup B} \in A$. Then by the hereditary property of the operator φ we have $i_{A \cup B} \in \varphi(A)$. By monotonicity (1) of the family $\pi(i,H)$ and the definition (2), we obtain the chain of inequalities

$$\begin{aligned} &F_\varphi(A \cup B) = \pi(i_{A \cup B}, A \cup B) \ge \pi(i_{A \cup B}, A) \ge \\ &\ge \min \pi(i,A) = F_\varphi(A) \ge \min\left(F_\varphi(A), F_\varphi(B)\right). \end{aligned}$$

This completes the proof of sufficiency.

Necessity. Consider a triple of the subset that satisfies (6). Assume that this triple violates the hereditary property, i.e.,

$$i \in A \cap \varphi(A \cup B), \text{ but } i \in \varphi(A).$$

Define a monotone system as follows:

$$\pi(i,A) = \pi(i,A \cup B) = 2,$$

$$\pi(j,A) = \pi(j,A \cup B) = 3, \; j \in A, \; j \ne i,$$

$$\pi(j,B) = 5 = \pi(j,A \cup B), \; j \in B - A.$$

On other subsets, we can easily define $\pi(i,H)$ without violating monotonicity.

In this case, we obviously have

$$F_\varphi(A \cup B) = 2 < \min\left(F_\varphi(A), F_\varphi(B)\right) = 3.$$

This contradiction proves necessity. ■

Proof of Theorem 3.

Necessity. We will show that "φ is a hereditary operator $\Rightarrow H^*$ is a solution of the maximization problem of $F_\varphi(H)$ on $\mathrm{B} - \mathrm{M}$." The proof relies on the following lemma.

LEMMA 1. The set of dead-end vertices M of the system (W, B) φ-generated by a hereditary operator consists of a single element. This element is the zero of the lower semilattice B.

Proof of Lemma 1. Assume that $H_1 \in \mathrm{B}$, $H_2 \in \mathrm{B}$, $H_1 \neq H_2$ and $\varphi(H_1) = \varphi(H_2) = \emptyset$. By Theorem 1, $H_1 \cap H_2 \in \mathrm{B}$. For definiteness let $H_1 \cap H_2 \neq H_1$. Consider sequence $(x_1, x_2, ..., x_k)$ of elements that are removed from the set W on passing from W to $H_1 \cap H_2$. Let x_j be the first element in this sequence such that $x_j \in H_1$. Then $x_1 \in \varphi(W - \{x_1, ..., x_{j-1}\})$ and $H_1 \subseteq W - W - \{x_1, ..., x_{j-1}\}$. By the hereditary property of the operator φ, this means that $x_1 \in \varphi(H_1)$. But this contradicts $\varphi(H_1) = \emptyset$. This proves the first assertion of the lemma. The second assertion is proved similarly.

Let us continue with the proof of necessity. Consider the element $x_1 \in H_1 = W$. Since φ is hereditary and $x_1 \in \varphi(H_1)$, for any $H \subseteq H_1$: $x_1 \in H$. From definition (2) we obtain

$$F_\varphi(H_1) = \pi(x_1, H_1) \geq \pi(x_1, H) \geq F_\varphi(H_2). \tag{A.1}$$

Relationship (A.1) implies that ether H_1 maximizes the function $F_\varphi(H)$ on $(W, \mathrm{B} - \mathrm{M})$ or x_1 is not an element of the maximizing set, i.e., the solution is the subset $H_1 - x_1 \in \mathrm{B}$. A similar argument can be applied to all H_i from the defining sequence. In the last step we obtain that ether some H_i is a solution of the problem or the subset $H_m - x_m$ is the solution. But, $\varphi(H_m - x_m) = \emptyset$ and therefore $H_m - x_m \notin \mathrm{B} - \mathrm{M}$.

Moreover, by Lemma 1, $H_m - x_m$ is the zero of the lower semilattice B, and therefor none of its subsets may belong to the system $(W, \mathrm{B} - \mathrm{M})$. Thus, the sought solution is contained in the defining sequence, which proves necessity.

Sufficiency. The proof is by contradiction. Let $X \subset Y$, $X, Y \in \mathrm{B}$, $x \in X \cap \varphi(Y)$, and $x \notin \varphi(X)$. Assume that W and Y are joined by a chain in (W, B): $W = H_1$, $H_2 = H_1 - x_1, ..., H_k - x_k = Y$. Consider the monotone system

1) $$\pi(y,H_i)=\begin{cases} i, & \text{if} \quad y=x_i, \\ k, & \text{if} \quad y\in X,\ i=\overline{1,k-1}, \\ k-1, & \text{if} \quad y\notin X,\ y\neq x_i; \end{cases}$$

2) $$\pi(y,H_k)=\begin{cases} 1, & \text{if} \quad y=x, \\ k, & \text{if} \quad y\in X,\ y\neq x, \\ k-1, & \text{if} \quad y\notin X; \end{cases}$$

3) $$\pi(y,Z)=0, \quad \text{if} \quad y\in Z\subseteq H_k-x;$$

4) $$\pi(y,X)=\begin{cases} k, & \text{if} \quad y\neq x, \\ 0 & \text{otherwize}; \end{cases}$$

5) $$\pi(y,Z)=\pi(y,H_i), \quad \text{if} \quad Z\subseteq H_{i-1},\ \mathrm{Z}\not\subset \mathrm{H}_{i+1},\ i=\overline{1,k};$$

6) $$\pi(y,Z)=0, \ \text{if} \quad Z\subset X;$$

7) $$\pi(y,Z)=\pi(y,H_k), \ \text{if} \quad X\subset Z\subset Y.$$

In this case we clearly have $F(X)=k$, where for any H from the defining sequence $(H_1,H_2,...,H_k,...)$ we have $F(H)<k$. This means that F-algorithm does not find the global maximum of the function (5).

Proof of the Theorem 4.

1) follows from quasiconcavity of the function $F_\varphi(H)$ (see Theorem 2).

2) Let H° and $\hat{H}$ be maximum-cardinality solutions of maximization problem for the function (5) and $H^\circ\neq\hat{H}$. Let H_k be the minimum set from the defining sequence such that $(H^\circ\cup\hat{H})\subseteq H_k$ and let x_k be the element corresponding to H_k. Clearly, $x_k\in H^\circ\cup\hat{H}$. Since φ is hereditary operator, it follows that $x_k\in\varphi(H^\circ\cup\hat{H})$ and ether $x_k\in\varphi(H^\circ)$ or $x_k\in\varphi(\hat{H})$. Let $x_k\in\varphi(H^\circ)$. Then by monotonicity of the system (W,π) and the definition of the function $F_\varphi(H)$ it follows that

$$F_\varphi(H_k)=\pi(x_k,H_k)\geq\pi(x_k,H^\circ)\geq F_\varphi(H^\circ).$$

This means that H_k is a solution of the maximization problem for (5). But $|H_k|>|H^\circ|=|\hat{H}|$, a contradiction. Parts 3) and 4) easily follow from 2).

Proof of the Theorem 5.

We will show that if (J^*,H^*) is a saddle point of the function $\Pi(J,H)$, then

$$F_\varphi(H^*) \ge F_\varphi(H),\ H \in \mathrm{B} - \mathrm{M}. \tag{A.2}$$

The image $\varphi(H)$ of any set $H \in \mathrm{B} - \mathrm{M}$ is nonempty and any sequence $J \in \Omega$ consists of all elements of the set $W - H_0$. Therefore it is easy to see that for any pair $J \in \Omega$, $H \in \mathrm{B} - \mathrm{M}$ there exists $x \in \varphi(H)$ such that $x = x_{J,H}$, and conversely by construction of a φ-generated system, for any $x \in \varphi(H)$ there is $J \in \Omega$ such that $x = x_{J,H}$. Therefore

$$\min_{J\in\Omega} \Pi(J,H) = \min_{x\in\varphi(H)} \pi(x,H).$$

Then we have

$$\Pi(J^*,H^*) = \max_{H\in \mathrm{B}-\mathrm{M}} \min_{J\in\Omega} \Pi(J,H) = \max_{H\in \mathrm{B}-\mathrm{M}} \min_{x\in\varphi(H)} \pi(x,H) = \max_{H\in \mathrm{B}-\mathrm{M}} F_\varphi(H).$$

On the other hand,

$$\Pi(J^*,H^*) = \min_{J\in\Omega} \Pi(J,H^*) = \min_{x\in\varphi(H^*)} \pi(x,H^*) = F_\varphi(H^*),$$

which proves (A.2).

We will now show that if H^* is a solution of the maximization problem for the function (5), then there exists J^* that satisfies (8).

The proof relies on the following lemma.

Let $J = (y_1, y_2, ..., y_m) \in \Omega$, $\mathrm{H}_J = \left\{H \middle| H = \{y_{k+1}, y_{k+2}, ..., y_m\} \cup H_0, k = \overline{1,m}\right\}$.

LEMMA 2.

$$\max_{H\in \mathrm{H}_J} \pi(J,H) = \max_{H\in \mathrm{B}-H_0} \Pi(J,H).$$

Proof of Lemma 2. Let $H \in \mathrm{B} - \mathrm{M}$, $H \in \mathrm{H}_J$. Then there exists $H' \in \mathrm{H}_J$, such that $H \subset H'$ for any $H'' \in \mathrm{H}_J$ such that $H'' \subset H'$ we have $H \not\subset H''$.

It is easy to see that by the hereditary property of the operator φ and the fact that $J \in \Omega$, we have $x_{J,H'} = x_{J,H}$. By monotonicity of the system $\langle W, \pi, F_\varphi \rangle$ this means that $\pi(x_{J,H'}, H') \ge \pi(x_{J,H}, H)$, i.e., $\Pi(J^*, H') \ge \Pi(J,H)$, which proves Lemma 2.

We now proceed to prove (8). From the definition of the functions $\Pi(J,H)$ and $F_\varphi(H)$ we have

$$\Pi(J^*,H^*) \ge F_\varphi(H). \tag{A.3}$$

On the other hand, by Lemma 2 and Theorem 3, we have H_{J^*}

$$F_{\varphi}(H^*) = \max_{H \in \mathrm{H}_{J^*}} \Pi(J^*, H) \geq \Pi(J^*, H^*). \tag{A.4}$$

Combining (A.3) and (A.4), we obtain

$$\max_{H \in \mathrm{H}_{J^*}} \Pi(J^*, H) = \Pi(J^*, H^*) = F_{\varphi}(H^*) = \min_{J \in \Omega} \Pi(J, H^*).$$

Again applying Lemma 2, we obtain inequality (8).

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