LOCAL TRANSFORMATIONS IN MONOTONIC SYSTEMS

II. Algorithms for Local Transformations of Monotonic Systems *

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Procedures are examined for establishing local changes to initial data that are necessary and sufficient to provide the required correction of the result of structuring: inclusion or exclusion of a specified element from the kernel of the monotonic system from the first part of this article.

1. Introduction

The problem of seeking small changes to initial data that result in a specified (desired) correction of the results of structuring was formulated in [1]. This problem is then posed and resolved in the framework of the theory of so-called monotonic systems, [1] in which the result of structuring the initial data is the kernel, or internal "central core" of the system under consideration, which is a type of subsystem that, in the sense of an exact solution of some extremal problem (solution algorithms for which are found in [2] and [3]), best reflects the "interrelations and interactions" of elements in the entire system. Necessary and sufficient conditions for the expansion (contraction) of the kernel of a system to an accuracy of one specified element ℓ were obtained in [1] for the class of p-monotonic systems presented there. At the same time the class of allowable changes to initial data was formulated in terms of local transformations of monotonic systems.

Previously, this problem (correcting the kernel of a monotonic system) was examined in [4] for two actual monotonic systems $\langle W,\pi_1\rangle$ and $\langle W,\pi_2\rangle$, where W is the set of objects, $|W| = N$, each element i of which is related to a subset y_i of the set indicators Y, while the functions $\pi_1(i,H)$ and $\pi_2(i,H)$ were defined as follows:

$$\pi_1(i,H) = \left| y_i \setminus Y^H \right|, \tag{1}$$

$$\pi_1(i,H) = \left| Y_H \setminus y_i \right|, \tag{2}$$

$$Y^H = \bigcap_{k\in H} y_k;\; Y_H = \bigcup_{k\in H} y_k. \tag{3}$$

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[1] Necessary definitions are provided in the appendix (see also [1-4]).

Sufficient conditions were obtained in [4] showing that expansion (contradiction) by only one subset y_ℓ permitted inclusion of one – specifically the ℓ-th – element into kernel G_1 of the first monotonic system or its exclusion from the kernel G_2 of the second system (or vice versa, respectively). At the same time the possibility of establishing the sought-after local transformation using these sufficient conditions is only mentioned in [4], while at the same time its practical implementation requires development of a special procedure.

Thus, the result of [4] was significantly broadened in [1] in two respects (instead of sufficient conditions for correction of the kernel for each of two actual monotonic systems, necessary and sufficient conditions are obtained for broad class of monotonic systems). However, development of a procedure for establishing the sought-after small change to initial data can not, in distinction from the actual theorems regarding necessary and sufficient conditions for kernel correction, be achieved in general terms for local transformations of monotonic systems that ignore the actual form of data representation. Thus, in the present work, which is actually a continuation not only of [1], but of [4] as well, construction of such a procedure is presented using the actual monotonic system $\langle W, \pi_1 \rangle$ as an example.

2. Local Transformations of Monotonic Systems on Boolean Data Matrices

Let $\Phi = \|\varphi_{ij}\|$ be a right-angled Boolean $N \times M$ matrix, each line of which corresponds to some objects i, which is an element of set W, $|W| = N$, and each column, to element j of the set of indicators Y, $|Y| = M$. In other words, matrix Φ defines a family W of subsets Y: $W = \{y_1, ..., y_N\}$, $y_i \subseteq Y$, where

$$\varphi_{ij} = \begin{cases} 1, & \text{if } j \in y_i, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

or conversely,

$$y_i = \{j \in Y, \varphi_{ij} = 1\},\ i = \overline{1, N}. \tag{5}$$

It is easy to see that functions π_1 and π_2, which are definable in (1)-(3) over the matrix Φ, exhibit monotonicity (see [A.1]).

Below we examine the problem of correcting the kernel G_1 of the monotonic system $\langle W,\pi_1\rangle$ (the subscript 1 may thus be dropped, writing $\langle W,\pi\rangle$, π, G, etc.). The problem of correcting kernel G_2 for monotonic system $\langle W,\pi_2\rangle$ is solved analogously.

Theorem 1. *The monotonic system $\langle W,\pi\rangle$, defined using (1) and (3), has separable variables (see [A.6]) and monotonic increments (see [A.7]).*

To prove this theorem, it is sufficient to express the function in the form

$$\pi(i,H)=\left|y_i \setminus Y^H\right|=\left|y_i\right|-\left|Y^H\right| \qquad (6)$$

and utilize the properties of set intersection.

Thus, Theorem 1 allows us to use the results of Theorem 5 and 6 from [1] regarding the expansion and contraction of the kernel of a p-monotonic system with monotonic increments in order to determine the necessary and sufficient conditions for correction of the kernel of a monotonic system.

We first define the concept of a local transformation of this monotonic system.

Theorem 2. *Expansion of set y_ℓ, i.e., the replacement of some zeros with ones in the ℓ-th row Φ_ℓ of matrix Φ, while leaving all other rows of the matrix unchanged is called a positive local transformation, while contraction of the set y_ℓ, i.e., the replacement of some ones by zeros in ℓ-th row Φ_ℓ of matrix Φ is called a negative local transformation of the monotonic system $\langle W,\pi\rangle$.*

The proof follows directly from the definition of positive and negative local transformations of a monotonic system with separable variables (see the appendix) and from the definition of system $\langle W,\pi\rangle$ (see [1], [3]).

Thus, a local transformation of a monotonic system $\langle W,\pi\rangle$ consists in the transition from initial data matrix Φ to a matrix [2] Φ', such that $y'_i = y_i$, $i=\overline{1,N}$, $i \neq \ell$ and $y'_\ell \supset y_\ell$ or $y'_\ell \subset y_\ell$.

[2] The prime denotes values, functions, and sets related to the transformed monotonic system

We denote by Δ_ℓ the set, and by n_ℓ, the number of added or removed elements in set y_ℓ:

$$\Delta_\ell = \begin{cases} y'_\ell \setminus \ell, & \text{if } y'_\ell \supset y_l, \\ y_l \setminus y'_\ell & \text{if } y'_\ell \subset y_l, \end{cases} \tag{7}$$

$$n_\ell = |\Delta_\ell| . \tag{8}$$

The number n_ℓ may be considered a parameter of a local transformation of a monotonic system that characterizes its "value," "power," etc.; as n_ℓ increases, the transformation becomes – in the framework of this approach – "less local."

Let us restate the conditions for correction of a kernel of a monotonic system relative to this characterizing number.

Theorem 3. *For the kernel of the monotonic system* $\langle W,\pi \rangle$ *on the transformed matrix* Φ' *, in order to obtain the set* $G' = G \cup \ell$ *via positive local transformation of the system* $\langle W,\pi \rangle$ *i.e., by expanding the number of ones in only* ℓ*-th row of this matrix,*

$$\left(y'_\ell \supset y_\ell \, , \; \Delta_\ell = y'_l \setminus y_\ell \, , \; n_\ell = |\Delta_\ell| = |y'_\ell \setminus y_l| ; \; y'_i = y_i \, , \; i = \overline{1,N} \, , \; i \neq \ell \right),$$

where $\ell \in W \setminus G$ *, it is necessary and sufficient that one of the following tree groups of conditions be satisfied*:

$$\left| y_{i_{m-1}} \right| - |y_i| < n_\ell < |y_g| - |y_\ell| , \tag{9}$$

$$n_l \geq |y_g| - |y_\ell| - \left| Y^G \setminus y_\ell \right| + \left| Y^G \cap \Delta_\ell \right| , \tag{10}$$

$$n_\ell > \pi(i_k, H_k) + \left| Y^{H_k} \setminus y_\ell \right| - \left| Y^{H_k} \cap \Delta_\ell \right| - \pi(\ell, G \cup \ell) + \left| Y^G \cap \Delta_\ell \right| , \; k = \overline{i(\Gamma_{p-1}), m-1} ; \tag{11}$$

$$n_\ell = |y_g| - |y_\ell| ; \tag{12}$$

$$n_\ell > |y_g| - |y_\ell| , \tag{13}$$

$$\left| \left(Y^{H_k} \setminus Y^G \right) \cap \Delta_\ell \right| \geq \pi(i_k, H_k) - \pi(g, G) + \left| \left(Y^{H_k} \setminus Y^G \right) \setminus y_\ell \right| , \; k = \overline{m, \mu} , \tag{14}$$

where

$$\mu : \left| y_{i_{\mu-1}} \right| < |y_\ell| + n_\ell \leq \left| y_{i_{\mu+1}} \right| . \tag{15}$$

Theorem 4. For the kernel of the monotonic system $\langle W,\pi' \rangle$ on the transformed matrix Φ' in order to obtain the set $G' = G \cup \ell$ via a negative local transformation of the system $\langle W,\pi \rangle$, i.e., by contracting the number of ones in only the ℓ-th row of this matrix

$$\left(y'_\ell \subset y_\ell \,,\ \Delta_\ell = y_l \setminus y'_\ell \,,\ n_\ell = |\Delta_\ell| = |y_\ell \setminus y'_l|\,;\ y'_i = y_i \,,\ i = \overline{1,N}\,,\ i \neq \ell\right),$$

where $\ell \in G$, it is necessary and sufficient that the following conditions be satisfied:

$$n_\ell > \pi(\ell,G) - F(G \setminus \ell) + \left|Y^G \cap \Delta_\ell\right|, \tag{16}$$

$$\pi(i_k,H_k) \leq F(G \setminus \ell),\ k = \overline{\lambda+1,N}\,, \tag{17}$$

$$\left|y_{\ell_k}\right| - \left|Y^{H_k \setminus \ell}\right| < F(G \setminus \ell),\ k = \overline{\mu,m-1}, \tag{18}$$

$$\pi(i_k,H_k) + \left|Y^{H_k \setminus \ell} \cap \Delta_\ell\right| < F(G \setminus \ell),\ k = \overline{1,\mu-1}, \tag{19}$$

where

$$F(G \setminus \ell) = \min_{i \in G \setminus \ell} |y_i| - \left|Y^{G \setminus \ell}\right|, \tag{20}$$

$$\mu :\ \left|y_{i_{\mu-1}}\right| < |y_\ell| - n_\ell \leq \left|y_{i_{\mu+1}}\right|. \tag{21}$$

Algorithms are presented below for construction of corresponding local transformations, one of which is based on using Theorem 3; the other, on Theorem 4. At the same time, two different problems are, generally speaking, actually solved each time:

1) finding the arbitrary local transformation that satisfies the necessary and sufficient conditions for a specified kernel correction;
2) finding the minimum value of n_ℓ among such transformations.

It turns out that for the case of kernel expansion $(G' = G \cup \ell)$ these are two problems are solved differently and with different complexity. In case of contraction $(G' = G \setminus \ell)$ the task of searching for an arbitrary local transformation and searching for the minimum local transformation that satisfies the necessary and sufficient conditions are of equal complexity, since they are performed using the same algorithm. [3]

[3] After a small altering using the same algorithm, we may formally find also the local transformations that have a maximum value n_ℓ among the satisfying conditions for kernel correction, although such a problem seems artificial to us. On the other hand, the search for a local transformation with a maximum number n_ℓ while preserving the kernel $G' = G$ is of general interest, although it falls outside the framework of this work.

3. Algorithms for Expanding and Contracting the Kernel of a Monotonic System

We examine a procedure for searching for a local transformation that satisfies the necessary and sufficient conditions for including a specified element ℓ in kernel G.

It is easy to see that such a transformation always exists, since there can always be found a number n_ℓ that satisfies the second group of conditions from Theorem 3. At the same time, no constraints are placed on the members of set $\Delta_\ell = y'_\ell \setminus y_\ell$ of ones added to the ℓ-th row of matrix Φ. Thus, in this case the procedure for constructing a local transformation that is necessary and sufficient for inclusion of element ℓ in the kernel reduces to the selection of n_ℓ, $n_\ell = |y_g| - |y_\ell|$ arbitrary elements of set $Y \setminus y_\ell$ and adding them to set y_ℓ.

At the same time, practice often requires finding a transformation that must change the minimum number n_ℓ of elements in matrix Φ, or at least, to establish whether the found local transformation is minimal in n_ℓ.

<u>Corollary to Theorem 3</u>. If $|y_{i_m}| - |y_{i_{m-1}}| = 1$ then the number n_ℓ that satisfied (12) is minimum. If $|y_{i_m}| - |y_{i_{m-1}}| > 1$ then the minimum allowable number n_ℓ satisfies either group I or group II of conditions of Theorem 3.

It is easy to see that in second case, the value of n_ℓ obtained from (12) may be larger than minimum.

Since the basis of the algorithms for searching for minimum transformations for expansion or contraction of the kernel coincide to a great degree, then while we are aware of an analogously simple algorithm for finding an arbitrary local transformation for contraction of the kernel, we examine the problem of contracting the kernel of a monotonic system.

We preface the immediate description of the algorithm with a number of corollaries to Theorem 4, which form the basis of its construction. References to conditions (16)-(19) everywhere denote the conditions of Theorem 4.

<u>LEMMA 1</u>. Satisfaction of conditions (17) and (18) are independent of the members of the set $\Delta_\ell = y_\ell \setminus y'_\ell$, and (17) is also independent of the number $n_\ell = |\Delta_\ell|$.

It follows that the first step of the algorithm for constructing the sought-after transformation of a monotonic system must be to verify that (17) is satisfied. For this, we first use Eq. (20) to calculate the value $F(G \setminus \ell)$. (Recall that the values of the function $\pi(i_k, H_k)$, $k = \overline{1,N}$ are known only after the kernel G is isolated in the initial monotonic system $\langle W, \pi \rangle$.

Then, as soon as n_ℓ is fixed, Eq. (21) is used to determine the associated new position of the element ℓ in the defining sequence I', i.e., the number μ, and condition (18) is verified.

LEMMA 2. In the conditions of Theorem 4, the number n_ℓ is constrained by the inequality

$$|y_\ell| - |y_g| < n_\ell \le |y_\ell|. \qquad (22)$$

Lemma 2 denotes that if (17) is satisfied, then the remaining part of the algorithm to search for transformation must represent a sequential verification of conditions (16), (18), and (19) for different values of n_ℓ, for example, the initial and smallest. In other words, the subsequent steps of the algorithm for construction of the sought-after transformation must, after verifying (17), be completed in an iterative procedure, in the beginning of which the defined value n_ℓ is set, for example, to one more that before (in this case, $n_\ell = |y_\ell| - |y_g|$ before the first iteration, and for the last, $n_\ell = |y_\ell|$). However, if we must find at least one local transformation that satisfies the necessary and sufficient conditions, then this cyclic procedure may terminate earlier, i.e., when conditions (16)-19) are satisfied for some n_ℓ. At the same time, as is easy to see, if the loop formally terminates, then it follows that (16)-(19) are not satisfied for any value of n_ℓ and the sought-after transformation does not exist.

LEMMA 3. If condition (18) is not satisfied for some $n_\ell = n_\ell^*$, then it is not satisfied for any $n_\ell > n_\ell^*$, as well.

Lemma 3 allows the search for local transformation to end for large values n_ℓ if it has been determined that the sought-after transformation does not exist in connection with non-satisfaction of (18).

LEMMA 4. If for some fixed n_ℓ condition (16) is satisfied for some set Δ_ℓ, $|\Delta_\ell| = n_\ell$, then it is also valid for any other set $\Delta_\ell^* \subseteq y_\ell$, $|\Delta_\ell| = n_\ell$ such that

$$|Y^G \cap \Delta_\ell^*| = |Y^G \cap \Delta_\ell|.$$

Lemma 4 allows us to simplify significantly the algorithm for constructing the sought-after local transformation of a monotonic system, since the required search of all subsets y_ℓ with the number of elements equal to n_ℓ are, in accordance with Lemma 4, replaced by a search of all allowable values $|Y^G \cap \Delta_\ell|$ entering into (16). In turn, each of its values corresponds to some family of subsets of set y_ℓ.

LEMMA 5. The range of values of $|Y^G \cap \Delta_\ell|$ that satisfy condition (16) is established by the following inequality:

$$\min_{\Delta_\ell} |Y^G \cap \Delta_\ell| \le |Y^G \cap \Delta_\ell| \le \max_{\Delta_\ell} |Y^G \cap \Delta_\ell|, \qquad (23)$$

where

$$\min_{\Delta_\ell} |Y^G \cap \Delta_\ell| = \begin{cases} 0, & \text{if } |y_\ell \setminus Y^G| \ge n_\ell, \\ n_\ell - |y_\ell \setminus Y^G| & \text{otherwise,} \end{cases} \qquad (24)$$

$$\max_{\Delta_\ell} |Y^G \cap \Delta_\ell| = n_\ell + F(G \setminus \ell) - \pi(\ell, G) - 1. \qquad (25)$$

Thus, condition (16) satisfies any set Δ_ℓ $\left(\Delta_\ell \subseteq y_\ell, |\Delta_\ell| = n_\ell\right)$ having value $|Y^G \cap \Delta_\ell|$ in the limits of the specified range.

COROLLARY. If $|y_\ell \setminus Y^G| \ge n_\ell$ and $\Delta_\ell \subseteq y_\ell \setminus Y^G$ or $|y_\ell \setminus Y^G| < n_\ell$ and $\Delta_\ell \supset y_\ell \setminus Y^G$ the (16) is satisfied.

Thus, in accordance with Lemma 5 in order to assure satisfaction of (16), it is sufficient to include in set Δ_ℓ as many elements of set $y_\ell \setminus Y^G$ as possible, since if $|y_\ell \setminus Y^G| < n_\ell$ then it is then sufficient to include the entire set $y_\ell \setminus Y^G$.

LEMMA 6. a) If $|y_\ell \setminus Y^G| \ge n_\ell$ and for some set Δ_ℓ^*, $Y^G \cap \Delta_\ell^* = \varnothing$ condition (19) is satisfied, then it is also satisfied for the set Δ_ℓ, $\Delta_\ell \subseteq y_\ell \setminus Y^G, |\Delta_\ell| = |\Delta_\ell^*| = n_\ell$. b) If $|y_\ell \setminus Y^G| < n_\ell$ and for set Δ_ℓ^*, $\left(y_\ell \setminus Y^G\right) \setminus \Delta_\ell^* = \varnothing$ condition (19) is satisfied, then it is also satisfied for the set Δ_ℓ, $\Delta_\ell \supset y_\ell \setminus Y^G$, $|\Delta_\ell| = |\Delta_\ell^*| = n_\ell$.

Thus, Lemmas 5 and 6 leave only the minimum value [4] from the range of values of quantities $\left|Y^G \cap \Delta_\ell\right|$ when constructing the sought-after set Δ_ℓ.

We denote the quantity $\left|Y^G \cap \Delta_\ell\right|$ by $n_\ell(m)$. [5]

It is easy to see that if $\left|y_\ell \setminus Y^G\right| \geq n_\ell$, then, in accordance with Lemmas 5 and 6, when searching for a local transformation to satisfy (16) and (19) we must take $n_\ell(m) = 0$, i.e., $\Delta_\ell \subseteq y_i \setminus Y^G$. If $\left|y_\ell \setminus Y^G\right| < n_\ell$ then $n_\ell(m) = n_\ell - \left|y_\ell \setminus Y^G\right|$, i.e., $\Delta_\ell \supset y_\ell \setminus Y^G$. In this way, (16) is guaranteed to be satisfied, and if (19) is not satisfied for such a value of quantity $n_\ell(m)$, then it is not satisfied for any other of its values.

<u>LEMMA 7</u>. If the inequality

$$\pi(i_k, H_k) < F(G \setminus \ell) \tag{26}$$

is not satisfied for at least one k, $k = \overline{1, \mu - 1}$, then for given value n_ℓ, the sought after local transformation does not exists.

Lemma 7 gives us a basis for terminating the search for set Δ_ℓ that satisfied conditions (16)-(19) for given n_ℓ, and go over to the next possible value of n_ℓ.

Now let inequality (26) be valid for all $k = \overline{1, \mu - 1}$. We denote by $\Delta_\ell(k)$ the set, and by $n_\ell(k) = \left|\Delta_\ell(k)\right|$ the number of elements of the set Y^{H_k}, $k = \overline{1, \mu}$, that are included in the set Δ_ℓ (i.e., $\underline{\Delta_\ell(k)} = Y^{H_k} \cap \Delta_\ell$), and for each $k = \overline{1, \mu - 1}$ we calculate the following quantity $n_\ell(k) = F(G \setminus \ell) - \pi(i_k, H_k)$.

<u>LEMMA 8</u>. If condition (19) is valid for some set Δ_ℓ, $\left|\Delta_\ell\right| = n_\ell$, then (19) is valid for any other set Δ_ℓ^*, $\left|\Delta_\ell^*\right| = n_\ell$, such that

$$n_\ell^*(k) = n_\ell(k) \;\; \forall\, k = \overline{1, \mu - 1}.$$

It is easy to see the analogy between the assertion of Lemma 4 relative to (16) and the assertion of Lemma 8 relative to (19).

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[4] If we must find all possible transformation that satisfy the necessary and sufficient conditions for excluding element ℓ from kernel G, then it is evident that in this case, we must examine all values of the quantity $\left|Y^G \cap \Delta_\ell\right|$ in range (18).

[5] The sense of such notation will be explained below.

Thus, just as Lemma 4 allowed us to go from a search of all subsets of set y_ℓ having a specified n_ℓ number of elements when constructing a set Δ_ℓ to satisfy (16), to a search of all possible values of $n_\ell(m) = \left|Y^G \cap \Delta_\ell\right|$. Lemma 8 allows us, instead of an analogous search when constructing a set Δ_ℓ that satisfies (19), to examine only all possible sets of values $n_\ell(k) = \left|Y^{H_k} \cap \Delta_\ell\right|$, $k = \overline{1,\mu-1}$. This, however, still cumbersome task is simplified significantly in light of the constraints imposed on these values.

LEMMA 9. The following relations are valid:

$$n_\ell(k) \le n_\ell(m) \;\; \forall\, k = \overline{1,\mu-1},$$

$$n_\ell(s) \le n_\ell(t) \;\; \forall\, s \le t,\; s,t = \overline{1,\mu-1}. \qquad (27)$$

It thus, follows that, having defined set Δ_ℓ by setting $n_\ell(k) = \left|Y^{H_k} \cap \Delta_\ell\right|$ and indexing k from μ to 1, we obtain for each following value $n_\ell(k)$ a more narrow range of possible values than before; in particular, if $n_\ell(m) = 0$, then $n_\ell(k) = 0$, $k = \overline{1,\mu-1}$.

LEMMA 10. If for set Δ_ℓ we have

$$n_\ell(k) < \overline{n_\ell(k)},\; k = \overline{1,\mu-1}, \qquad (28)$$

where $n_\ell(k) = \left|Y^{H_k} \cap \Delta_\ell\right|$ and $\overline{n_\ell(k)} = F(G \setminus \ell) - \pi(i_k, H_k) > 0$, then for this set Δ_ℓ, condition (19) is satisfied.

COROLLARY. If condition (26) is valid for all $k = \overline{1,\mu-1}$ and in addition, $n_\ell(k) = 0$, $k = \overline{1,\mu-1}$, then for this Δ_ℓ, condition (19) is satisfied.

LEMMA 11. a) If $\left|Y^G \setminus Y^{H_{\mu-1}}\right| \ge n_\ell(m)$ and for some set Δ_ℓ^*, $\Delta_\ell^* \cap Y^{H_{\mu-1}} \ne \varnothing$ condition (19) is satisfied, then it is also satisfied for set Δ_ℓ, $\Delta_\ell(m) \subseteq Y^G \setminus Y^{H_{\mu-1}}$, $\left|\Delta_\ell(m)\right| = \left|\Delta_\ell \cap Y^G\right| = \left|\Delta_\ell^* \cap Y^G\right| = n_\ell(m)$, $\left|\Delta_\ell\right| = \left|\Delta_\ell^*\right| = n_\ell$. b) If $\left|Y^G \setminus Y^{H_{\mu-1}}\right| < n_\ell(m)$ and for some set Δ_ℓ, $\left(Y^G \setminus Y^{H_{\mu-1}}\right) \setminus \Delta_\ell^* \ne \varnothing$ condition (19) is satisfied, then it is also satisfied for set Δ_ℓ, $\Delta_\ell \supset Y^G \setminus Y^{H_{\mu-1}}$, $\left|\Delta_\ell(m)\right| = \left|\Delta_\ell \cap Y^G\right| = \left|\Delta_\ell^* \cap Y^G\right| = n_\ell(m)$, $\left|\Delta_\ell\right| = \left|\Delta_\ell^*\right| = n_\ell$.

In order to preserve uniqueness of expressions that comprise analogous assertions for $k = \overline{1,\mu-1}$ and for $k = 1$, we set $Y^{H_0} = \varnothing$.

LEMMA 12. a) If $\left|Y^{H_k} \setminus Y^{H_{k-1}}\right| \geq n_\ell(k)$ for some k, $k=\overline{1,\mu-1}$ and for some set Δ_ℓ^*, $\Delta_\ell^* \cap Y^{H_{k-1}} \neq \varnothing$, for this k the inequality in (19) is satisfied, then it is also satisfied for the set Δ_ℓ, $\Delta_\ell(k) \subseteq Y^{H_k} \setminus Y^{H_{k-1}}$, $\left|\Delta_\ell(k)\right| = \left|\Delta_\ell \cap Y^{H_k}\right| = \left|\Delta_\ell^* \cap Y^{H_k}\right| = n_\ell(k)$, $\left|\Delta_\ell\right| = \left|\Delta_\ell^*\right| = n_\ell$. b) If $\left|Y^{H_k} \setminus Y^{H_{k-1}}\right| < n_\ell(k)$ for some k, $k=\overline{1,\mu-1}$ and for some set Δ_ℓ^*, $\left(Y^{H_k} \setminus Y^{H_{k-1}}\right) \setminus \Delta_\ell^* \neq \varnothing$, for this k the inequality (19) is satisfied, then it is also satisfied for the set Δ_ℓ, $\Delta_\ell \supset \Delta_\ell(k) \supset Y^{H_k} \setminus Y^{H_{k-1}}$, $\left|\Delta_\ell(k)\right| = \left|\Delta_\ell \cap Y^{H_k}\right| = \left|\Delta_\ell^* \cap Y^{H_k}\right| = n_\ell(k)$, $\left|\Delta_\ell\right| = \left|\Delta_\ell^*\right| = n_\ell$.

It is easy to note the analogy between the assertions of Lemma 6 relative to the advantageous inclusion of elements from set $y_\ell \setminus Y^G$ into the set Δ_ℓ in order to satisfy (19), and those of Lemmas 11 and 12.

Lemmas 5 (corollaries), 6, 10, 11 and 12 permit the organization of a sequential uninterrupted construction of a set Δ_ℓ that satisfies (19).

$n_\ell(\mu-1)$

First, in the case where $\left|y_\ell \setminus Y^G\right| < n_\ell$ all elements of set Δ_ℓ must be included in the set $y_\ell \setminus Y^G$ in accordance with Lemmas 5 and 6. In the opposite case, we may select any set $\Delta_\ell \subseteq y_\ell \setminus Y^G$, and if condition (26) is thereby satisfied for $k=\overline{1,\mu-1}$, then construction of sought-after local transformation is stopped. The given value of n_ℓ and the resulting set Δ_ℓ, $\Delta_\ell \subseteq y_\ell$, $\left|\Delta_\ell\right| = n_\ell$ satisfy the necessary and sufficient conditions foe excluding element ℓ from kernel G. Further, if $\left|Y^G \setminus Y^{H_{\mu-1}}\right| \geq n_\ell(m) = n_\ell - \left|y_\ell \setminus Y^G\right|$ then set Δ_ℓ includes $n_\ell(m)$ arbitrary elements of set $Y^G \setminus Y^{H_{\mu-1}}$, which form the set $\Delta_\ell(m)$. Then, if conditions (26), $k=\overline{1,\mu-1}$ are satisfied at the same time, then construction of set Δ_ℓ is also stopped; specifically, a set $\Delta_\ell = \left(y_\ell \setminus Y^G\right) \cup \Delta_\ell(m)$ is selected. If $\left|Y^G \setminus Y^{H_{\mu-1}}\right| < n_\ell(m) = n_\ell - \left|y_\ell \setminus Y^G\right|$ then set $Y^G \setminus Y^{H_{\mu-1}}$ is included in Δ_ℓ entirely: $Y^G \setminus Y^{H_{\mu-1}} \subset \Delta_\ell(m) \subset \Delta_\ell$. Then the number of elements remaining to be included in Δ_ℓ is $n_\ell(\mu-1) = n_\ell(m) - \left|Y^G \setminus Y^{H_{\mu-1}}\right|$. If $n_\ell(\mu-1) \leq \overline{n_\ell(\mu-1)}$, which assures satisfaction of the corresponding inequality (19), then we examine the following set $Y^{H_{\mu-1}} \setminus Y^{H_{\mu-2}}$. If $\left|Y^{H_{\mu-1}} \setminus Y^{H_{\mu-2}}\right| \geq n_\ell(\mu-1)$, then in Δ_ℓ we include $n_\ell(\mu-1)$ arbitrary elements of this set, and conclude the construction of the sought-after transformation.

If $\left|Y^{H_{\mu-1}} \setminus Y^{H_{\mu-2}}\right| < n_\ell(\mu-1)$ then set $Y^{H_{\mu-1}} \setminus Y^{H_{\mu-2}}$ is included entirely in Δ_ℓ, etc.

This last comment before presenting the description of the algorithm is related to the arbitrary nature of the selection of subsets of some set with a given number of elements. To have unique definiteness in the steps of the algorithm, we must have arbitrary, yet fixed way of executing these operations. We shall agree, for example, that in such situations for inclusion into Δ_ℓ we shall select the required number of elements from the specified set having the smallest number in its initial list of elements.

Algorithm for constructing local transformation for
exclusion of element ℓ from Kernel G.

The initial data for operation of the algorithm are a Boolean $N \times M$ matrix Φ, a kernel G, $G \subseteq W$, a monotonic system $\langle W, \pi \rangle$, and element ℓ, $\ell \in G$, which must be excluded from the kernel by replacing ones by zeros only in ℓ-th row of Φ. The sets H_k and Y^{H_k}, $k = \overline{1,N}$, are also assumed known, as are quantities $\pi(i_k, H_k)$, $k = \overline{1,N}$, defined in the process of isolating kernel G.

The algorithm is described in steps.

1. The quantity $F(G \setminus \ell)$ is calculated (from [20]).
2. Condition (17) is verified. If it is not satisfied, go to step 22.
3. An allowable value is assigned to n_ℓ. This is done using a loop in $n_\ell = \overline{|y_\ell| - |y_g| + 1, |y_\ell|}$; in each iteration, $n_\ell := n_\ell + 1$. The last step of the algorithm included in this loop is 21.
4. The number μ is calculated (from [21]).
5. Condition (18) is verified. If it is not satisfied, go to step 22.
6. Condition (26) is verified for $k = \overline{1, \mu - 1}$. If it is not satisfied foe even one k, $k = \overline{1, \mu - 1}$, go to step 21.
7. The following inequality is verified:

   $$\pi(\ell, G) = |y_\ell \setminus Y^G| \geq n_\ell .$$

   If it is not valid, then go to step 9.
8. Select a set $\Delta_\ell \subseteq y_\ell \setminus Y^G$, $|\Delta_\ell| = n_\ell$, here $n_\ell(m) = |Y^G \cap \Delta_\ell| := 0$. The result is n_ℓ, Δ_ℓ. Stop.
9. Calculate the quantity $n_\ell(m) := n_\ell - |y_\ell \setminus Y^G|$.

10. The following inequality is verified:

$$\left|Y^{G} \setminus Y^{H_{\mu-1}}\right| \geq n_{\ell}(m).$$

If it is not valid, then go to step 13.

11. Select a set $\Delta_{\ell}(m) \subseteq Y^{G} \setminus Y^{H_{\mu-1}}$, $\left|\Delta_{\ell}(m)\right| = n_{\ell}(m)$.

12. Result is n_{ℓ}, $\Delta_{\ell} := \left(y_{\ell} \setminus Y^{G}\right) \cup \Delta_{\ell}(m)$. Stop.

13. Take $Y^{H_0} = \varnothing$.

14. Initialize a loop in $k = \overline{\mu - 1, 1}$ (decrementing after each step).

15. Calculate the quantity

$$n_{\ell}(k) = \begin{cases} n_{\ell}(m) - \left|Y^{G} \setminus Y^{H_{\mu-1}}\right|, & k = \mu - 1, \\ n_{\ell}(k+1) - \left|Y^{H_{k+1}} \setminus Y^{H_k}\right|, & k < \mu - 1. \end{cases}$$

16. The following inequality is verified:

$$n_{\ell}(k) < n_{\ell}(k) = F(G \setminus \ell) - \pi(i_k, H_k).$$

If it is not valid, then go to step 21.

17. The following inequality is verified:

$$\left|Y^{H_k} \setminus Y^{H_{k-1}}\right| \geq n_{\ell}(k).$$

If it is not valid, then go to step 19.

18. If the loop in $k = \overline{\mu - 1, 1}$ is not complete, i.e., $k > 1$, then the value k is decremented and the algorithm proceeds to step 15 (after completing this loop, go to stem 21).

19. Select a set $\Delta_{\ell}(k) \subseteq Y^{H_k} \setminus Y^{H_{k-1}}$, $\left|\Delta_{\ell}(k) = n_{\ell}(k)\right|$.

20. Result is n_{ℓ}, $\Delta_{\ell} := \left(y_{\ell} \setminus Y^{G}\right) \cup \left(Y^{G} \setminus Y^{H_k}\right) \cup \Delta_{\ell}(k)$. Stop.

21. (For a given value of n_{ℓ}, the sought-after transformation does not exist.) If the loop in $n_{\ell} = \overline{|y_{\ell}| - |y_g| + 1, |y_{\ell}|}$ is not complete, i.e., $n_{\ell} < |y_{\ell}|$, then increment $n_{\ell} := n_{\ell} + 1$ and go to step 4.

22. Result is that the sought-after local transformation does not exist. Stop.

Theorem 5. If there exists at least one negative local transformation that satisfies the necessary and sufficient conditions of the kernel $(G' = G \setminus \ell)$ of a monotonic system, then a transformation having a minimum characteristic number n_{ℓ} among such transformations will be the result of applying the algorithm described above.

Proof of this theorem, based on the sequential use of assertions of Lemmas 1-12, and the proof of the lemmas themselves are omitted, since they are only of technical interest.

APPENDIX

A monotonic system $\langle W,\pi \rangle$ is defined as a finite set W, $|W| = N$, with a specified numerical (weighted) function $\pi(i,H)$, $i \in H$, $H \subseteq W$ exhibiting monotonicity, i.e.,

$$\pi(i,H \setminus j) \le \pi(i,H) \ \forall i \in H \setminus j, \ \forall j \in H, \ \forall H \subseteq W. \qquad \text{(A.1)}$$

The kernel of a monotonic system $\langle W,\pi \rangle$ is a set $G \subseteq W$ such that

$$F(H) < F(G) \ \forall H \supset G, \qquad \text{(A.2)}$$

$$F(H) \le F(G) \ \forall H \subseteq G, \qquad \text{(A.3)}$$

where the function F is defined as follows:

$$F(H) = \min_{i \in H} \pi(i,H) \ \forall H \subseteq W. \qquad \text{(A.4)}$$

The algorithm for isolating the kernel of a monotonic system [3] is based on construction of a special (nonrepeating) sequence $I = i_1,...,i_N$ of elements of set W, called defining, via the sequential selection of element i_k with minimum weight

$$\pi(i_k,H_k) = \min_{i \in H_k} \pi(i,H_k) \qquad \text{(A.5)}$$

among the sets H_k of elements remaining at the k-th step, and its inclusion in the k-th position of a sequence, after which the weight of the other elements are recalculated for the new set of remaining elements $H_{k+1} = H_k \setminus i_k$. The sets $\langle H_k, k = \overline{1,N} \rangle$ thus form the sequence $\overline{H} = H_1,...,H_N$, that is "parallel" to the defining sequence. The kernel G is then isolated by verifying conditions (A.2) and (A.3) not only among the sets of sequence $\overline{H}$. The number of the kernel in the sequence $\overline{H}$ is denoted by m: $G = H_m$. It is the number of the first element of kernel G in sequence I: $g = i_m$.

Among the sets $\langle H_k, k = \overline{1,N} \rangle$ there is a subsequence of sets $\overline{\Gamma} = \langle \Gamma_j, j = \overline{1,p} \rangle$ where $p \le N$ for which

$$F(H) < F(\Gamma_j) \ \forall H \supset \Gamma_j,$$

$$F(H) \le F(\Gamma_j) \ \forall H, \Gamma_j \supset H \supset \Gamma_{j+1}.$$

The latter set Γ_p in sequence $\overline{\Gamma}$ of quasikernels is the kernel $G = \Gamma_p$. We denoted by $i(\Gamma_{p-1})$ the number of the first element of set Γ_{p-1} in sequence I.

The monotonic system $\langle W,\pi \rangle$ is called a system with separable variables of

$$\pi(i,H) = p(i) + r(H). \tag{A.6}$$

The monotonic system $\langle W,\pi \rangle$ has monotonic increments if

$$\pi(i,H) - \pi(i,H \setminus \ell) \le \pi(i,E) - \pi(i,E \setminus \ell) \;\; \forall\, i,\ell \in E,\; i \ne \ell,\; \forall\, E \subseteq H,\; \forall\, H \subseteq W. \tag{A.7}$$

Local transformations of a monotonic system with separable variables is defined by the relations

$$\begin{aligned} p'(i) &= p(i) \quad \forall\, i \in W, \quad i \ne \ell, \\ r'(H) &= r(H) \;\; \forall\, H \subset W, \;\; \ell \notin H, \end{aligned} \tag{A.8}$$

and it is positive if

$$\begin{aligned} p'(\ell) &> p(\ell), \\ r'(H) &\le r(H) \;\; \forall\, H \subseteq W, \;\; \ell \in H, \end{aligned} \tag{A.9}$$

and negative if

$$\begin{aligned} p'(\ell) &< p(\ell), \\ r'(H) &\ge r(H) \;\; \forall\, H \subseteq W, \;\; \ell \in H. \end{aligned} \tag{A.10}$$

We denote by λ and μ the number of element ℓ relative to sequence I before transformation $(i_\lambda = \ell)$ and to sequence I' after transformation $(j_\mu = \ell)$. It is shown that when (A.8) and (A.9), $\mu \ge \lambda$, and the mutual position of remaining elements in the defining sequences of monotonic systems with separable variables do not change.

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