LOCAL TRANSFORMATIONS IN MONOTONIC SYSTEMS

I. Correcting the kernel in Monotonic Systems *

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An exact solution is presented for the problem of finding local changes in initial data necessary and sufficient to provide desired correction of the results of structuring: inclusion or exclusion of a specified element from the kernel of a monotonic system [1]. With the goal, a new class p -monotonic systems is presented and several of their properties are examined; in particular, those that permit solution of the direct structuring problem – isolating the kernel – using an algorithm that is an order faster.

1. Introduction

The following two difficulties have been identified as a measure of the experience in applying methods of structured analysis to complex systems (see, e.g., [2,3]).

1. The search character of the problem of finding the best partition of the initial set of object according to one or another criterion, which leads to a need to be satisfied with local extrema of the corresponding criterion when solving applied problems.
2. The absence of possibilities for evaluating the stability of the result of structuring for small changes of initial data, as well as solving the inverse problem: to find changes in initial data that are necessary and/or sufficient to obtain desired correction of the result of structuring. Experts normally easily determine the need and proper direction of such correction by considering additional a priori information about the system under study.

In fact, the inverse problem is more a problem not of analysis, but synthesis; its solution answers the question: what must the elements of a system be in order for it to exhibit the required structural properties? For example, when studying organizational systems whose elements are positions of responsibility, solution of the inverse problem shows how the functions of different positions must be changed in order to improve organizational structure.

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A new method of structuring that depends on construction of so-called monotonic systems is proposed to solve the first of the problems presented in [1]. Using such a construction, we may naturally define the concept of “internal core” or “kernel” of the system under study, which is one of its subsystems such that best reflects “interaction and interrelation” of elements in the entire system in the sense of a precise solution of some extremal problem.

The essence of this method consists in the following. Using a set of all subsets $\{H\}$ of the initial set W of elements, we construct a system of numerical functions from the elements $\pi(i,H)$, $i \in H$, $H \subseteq W$ which exhibit monotonicity

$$\pi(i,H \setminus j) \leq \pi(i,H) \ \forall\, i \in H \setminus j\,,\ \forall\, H \subseteq W\,. \tag{1}$$

We use the binary relation between the elements of the initial set, represented in the form of a graph, and the numerical matrix of relations between elements, which is a right-angle “object-characteristic” matrix that relates each element $i \in W$ to some subset y_i of another set Y of binary attributes [1, 4-6], etc., to specify these functions. Then the problem of structuring the monotonic system $\langle W,\pi \rangle$ (i.e., the set of objects W with a function π specified “on it” that satisfies (1) is posed as a formal problem of identifying the largest subset G, $G \subseteq W$, called the kernel, such that the function $F(H)$ attains a maximum, i.e.,

$$F(G) = \max_{H \subseteq W} F(H)\,, \tag{2}$$

where

$$F(H) = \min_{i \in H} \pi(i,H)\,.$$

For solution of this problem, efficient algorithms having a complexity of the order $N^2/2$, where $N = |W|$, were presented in [1,4]. We note that the global maximum of the corresponding criterion is found using these algorithms.

Further development if this method for structuring has been related primarily to the development of actual systems of monotonic functions, which allow us to make a simple and important interpretation of the found kernel, from a practical point of view [4-8].

Let, for example, $W = \{x_1, x_2, ..., x_N\}$ be a finite set of positive numbers, and

$$\pi(i,H) = a^{x_i} + \sum_{x_j \in H} a^{x_j} \tag{3}$$

for any $a > 1$.

Then the solution of problem (2) to find a kernel may be viewed as a formal division of the initial set of numbers into “large” (kernel elements) and “small”.

In [1-4] the function $\pi(i,H)=\sum_{j\in H} a_{ij}$ is examined, specified for a matrix of relations $A=\|a_{ij}\|$, $i,j\in W$. The kernel of this system is interpreted as the subset of boundary, more remote elements of the set W. Two monotonic systems were constructed in [5] using a boolean right-angle matrix that describes the distribution of the control function in organizational system:

$$\pi_1(i,H)=\left| y_i \setminus \left(\bigcap_{k\in H} y_k \right) \right|, \tag{4}$$

$$\pi_2(i,H)=\left| \left(\bigcup_{k\in H} y_k \right) \setminus y_i \right|, \tag{5}$$

where y_i is the set of control functions entering into the sphere of competition of the i-th position. The kernels G_1 and G_2 are thus interpreted as, respectively, the subset of positions, i.e., coordinators of control actions in organizational system, and the subset of controllers, i.e., executives (specialists).[1]

Various monotonic systems were examined on graphs in [6] and [7]. In [6], W was a set of graph nodes, and as a result a subgraph $G\subseteq W$ of the initial graph was found, comprised of the “most saturated” subgraphs isomorphic to some specified small graph Γ (for example, a “triangle with loops”). In [7], the concept of neighborhood (the set of adjacent vertices) was used for each vertex in the initial and small graphs. The monotonic system itself was constructed from elements of matrix Φ of the relation between vertices of the small and large graph. The function $\pi((i,p),H)$, where i is a vertex number in the large graph, and p, that in a small, is determined as the sum of estimates of accuracy and fullness (i.e., the powers of the projection of the neighborhoods of each vertex of the large graph into the small, and visa versa). The kernel $G\subseteq\Phi=W$, viewed as a set of ones in matrix Φ, uniquely determines the optimum relation between vertices of the large and small graph.

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[1] More detail about this example is found in the second half of this article.

In [8], the original matrix of relations $A = \| a_{ij} \|$ between objects is first transformed into matrix $P = \| p_{ij} \|$ of transition probabilities between states of some homogeneous Markov chain [9]. A monotonic system is then constructed using the set of arcs of a corresponding transition graph, and the quantity $\pi((i, j), H)$, where $H \subseteq W = \{(i, j), i, j = \overline{1, N}\}$, is calculated as the average number of transitions from state i to j on the constructed Markov chain if transactions are forbidden along the arcs of the set $W \setminus H$. In the kernel G of this monotonic system, viewed as some graph, we can find components of connectedness, the set of vertices of which from classes of strongly connected objects (taking into account not only of direct, as in [4], but of mediated relations as well).

Thus, the apparatus of monotonic systems allows efficient solution of a large number of different structuring problems.

In addition, as is shown in this work, the method of monotonic systems may be used also for removal of the second noted difficulty in applying formal methods of structuring.

Until now solution of this problem in practice required repeated solution of the same "direct" structuring problem for some compiled number of modifications of initial data and/or for variation in algorithm parameters. Only in this way can we estimate the stability of the obtained structure, and similarly, to lesser extent, the possibility of correcting the initial data in order to attain the desired structuring result. It is evident that search like this, where there is a large number of allowable small changes to initial data, is necessarily constrained. This does not allow us to consider the results of such an experimental estimate of structure stability and the possibilities of correcting it to be sufficiently reliable.

At the same time, the theory of monotonic systems allows us, within limits, to "predict" the changes of the structuring results for one or another change in initial data (i.e., to yield a partial solution of the reverse problem), without direct solution of the “direct” structuring problem. This problem – the dependence of the structuring results, i.e., the makeup of the kernel, or small changes in initial data – was formulated in [5]. Specifically, in this work, we formulate the problem of correcting the kernel G of one specified object, and sufficient conditions that most satisfy the sought-after local transformation of initial data are proposed.

However, the solution of this problem proposed in [5] suffers from a number of shortcomings.

First, practically, the search for initial data that satisfies the sufficient conditions for a specified range in the kernel often goes beyond what may be considered to be a “small” changes. Thus, theorems of sufficient conditions give no guarantee that the same results cannot be obtained for small changes in the initial data.

Second, sufficient conditions for a specified transformation were examined in [5] only for two actual monotonic systems, while at the same time, practice requires solution of an analogous problem in a more general case.

Third, the possibility of constructing the sought-after local transformation based on examination of sufficient conditions is only mentioned in [5], while its practical implementation requires development of a constructive procedure for such construction.

In this work, we try to overcome these shortcomings. At the same time, we examine a class of monotonic systems that allow us to solve the problem of isolating the kernel in the general case is not $N^2/2$, but in N steps [10]. For monotonic systems in this class, we introduce the strict understanding of a local transformation and propose necessary and sufficient conditions for the specified kernel correction. Finally, in the second part of the article, we construct a constructive procedure for searching for a local transformation that is in the specified sense “minimal” in the given class of transformations.

2. p -Monotonic Systems

Isolation of the kernel of the monotonic system in [4] requires construction of a sequence $I = \langle i_1,...,i_N \rangle$ of elements W, that [2]

$$\pi(i_k ,H_k) = \min_{i \in H_k} \pi(i,H_k), \tag{6}$$

where

$$H_1 = W\ ,\ H_2 \setminus i_1,..., H_k = H_{k-1} \setminus i_{k-1},...,\ k = \overline{1,N}\ . \tag{7}$$

(The sequence I is called defining [1]).

We the find a set $G = H_m$ in the sequence $\overline{H} = \langle H_1,...,H_N \rangle$, for which the following is valid:

$$F(H_k) < F(G)\ \ \forall\, H_k \supset G\ \ \left(\forall\, k = \overline{1,m-1}\right), \tag{8}$$

$$F(H_k) \le F(G)\ \ \forall\, H_k \subseteq G\ \ \left(\forall\, k = \overline{m,N}\right). \tag{9}$$

It was proved in [1] that this set is the kernel of a monotonic system.

A class of p-monotonic systems is isolated in [10]. For the finite set W let there be specified a nonstrict linear order P, which naturally sorts all elements of set W into the sequence $I^P = \langle i_1^P,...,i_N^P \rangle$, where $\left(i_s^P ,i_t^P\right) \in P$ for $s \le t$, with an accuracy to p-even elements [3] (elements $x,y \in W$ are called p-even if $(x,y) \in P$ and $(y,x) \in P$ simultaneously).

Definition. The monotonic system $\langle W,\pi \rangle$ will be called p-monotonic if there is an order P in W such, that for any subset H, $H \subseteq W$ the following is true:

$$\pi(i,H) < \pi(j,H)\ \ \forall\, (i,j) \in P\ ,\ (j,i) \notin P\ , \tag{10}$$

$$\pi(i,H) = \pi(j,H)\ \ \forall\, (i,j) \in P\ ,\ (j,i) \in P\ . \tag{11}$$

Among the examples of actual monotonic systems presented above that are related to this class are systems (4) and (5) from [5], and system (3).

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[2] If at some step in the algorithm the selection of element i_k is not unique, then an element that has a minimum number in the initial list of elements of set W is selected. The defining sequence, constructed using this rule, is called fixed and elements i_k and i_s, for which $\pi(i_k H_k) = \pi(i_s ,H_k)$ is valid, are I -even.

[3] Analogously, we henceforth will consider the fixed sequence I^P

The specification of order P in W apart from function $\pi(i,H)$ and the definition of the class of p-monotonic systems allows formulation of important properties of such monotonic systems and to propose a significantly simpler and faster algorithm for constructing a defining sequence and isolation of the kernel.

Theorem 1. *If $\langle W,\pi \rangle$ is a p-monotonic system then the following relations are valid:*

$$\pi(i,H) < \pi(j,H) \Leftrightarrow \pi(i,W) < \pi(j,W) \ \ \forall\, i,j \in H\,,\ \forall\, H \subseteq W\,, \tag{12}$$

$$\pi(i,H) = \pi(j,H) \Leftrightarrow \pi(i,W) = \pi(j,W) \ \ \forall\, i,j \in H\,,\ \forall\, H \subseteq W\,. \tag{13}$$

In other words, [4] if $\langle W,\pi \rangle$ is a p-monotonic system, then order P is defined by values of function $\pi(i,W)$ on set W.

Theorem 2 [10]. *The defining sequence I of p-monotonic system coincides with the sequence I^P corresponding to order P.*

Corollary. [5] *The algorithm for isolating the kernel of p-monotonic system reduces to calculation of the function $\pi(i_k,H_k)$, $i_k = i_k^P$, $H_k = H_k^P$, $k = \overline{1,N}$, on the sequence I^P; the set G is then selected to serve as the kernel G, satisfying (8) and (9).*

Thus, the algorithm for isolating the kernel of a p-monotonic system consists of two independent stages. First, construct a sequence I^P, corresponding to the specified order P, which in light of Theorem 2 is defined as $I^P = I = \langle i_1,\ldots,i_N \rangle$. Second, calculate the values of $\pi(i_k,H_k)$, $k = \overline{1,N}$ and select the set $G = H_m$ that satisfies (8) and (9).

The advantage of this algorithm over those in [1-4] consists in that that defining sequence is constructed here without calculating $\pi(i,H)$ for all elements of each set H_k and without finding the minimum of this function at each step. Thus, if we consider the calculations of the values $\pi(i,H)$, $i \in H$, $H \subseteq W$ to be limiting (the most cumbersome) operation in the algorithm for isolating the kernel of a monotonic system, then the complexity of defining kernel G using this algorithm is proportional to N, while that for the general algorithm is $N^2/2$.

We now isolate a subclass of the class of p-monotonic systems.

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4 In the formulation of Theorem 2 we have in mind fixed sequences I and I^P; in the opposite case, we must add the phrase “to an accuracy of transpositions of p-even and I-even elements.”

5 Proof of all assertions are presented in the Appendix.

Definition. A monotonic system on a finite set W with a function $\pi(i,H)$ defined on it is called with separable variables on it and satisfying the relations

$$\pi(i,H) = p(i) + r(H), \tag{14}$$

$$r(E) \le r(H) \;\; \forall\, E,H \subseteq W,\; E \subseteq H. \tag{15}$$

In other words, function π in this case is the sum of two separate functions, one of which depends only on i, the other, only on H; the second is monotonic.

The above system (3), (4), and (5) relate specifically to this subclass of p-monotonic systems.

Theorem 3. *The monotonic system $\langle W,\pi \rangle$ with separable variables is p-monotonic and its order is determined by the sequence I^P of elements of set W, ordered by the increase (not decrease) in $p(i)$, i.e.,*

$$I^P = \langle i_1^P,\dots,i_N^P \rangle :\; p(i_s^P) \le p(i_t^P) \;\; \forall\, s < t.$$

Thus, to isolate the kernel of a monotonic system with separable variables, we may use an algorithm for isolating the kernel of p-monotonic systems described above. Here, in the first stage of the algorithm all elements of set W are ordered by the increase (not decrease) in function $p(i)$, then the values of function $r(H)$ are calculated for sets H_k, $k = \overline{1,N}$, and function $\pi(i_k,H_k) = p(i_k) + r(H_k)$, $k = \overline{1,N}$. The kernel is selected using (8) and (9).

At the same time not all p-monotonic systems satisfy the separability condition (14). An example of such a system is the system $\langle W,\pi \rangle$, where

$$\pi(i,H) = [r(H)]^{p(i)},\; p(i) \ge 0,\; \forall\, i \in W;\; r(W) \ge 1 \;\; \forall\, H \subseteq W,$$

and the function $r(H)$ satisfies (15).

Finally, we isolate yet another class of monotonic systems.

Definition. We will call the monotonic system $\langle W,\pi\rangle$ a system with monotonic increments if for any arbitrary element ℓ and for any sets E and H, $E,H \subseteq W$ such that $E \subseteq H$, $\ell \notin H$, the following relations are satisfied:

$$\pi(i,H\cup\ell)-\pi(i,H)\le\pi(i,E\cup\ell)-\pi(i,E) \ \ \forall\, i\in E\,. \tag{16}$$

It may be shown that systems (4) and (5), as well as the matrix of relations from [4], are related to this class of system. The latter, however, is not p-monotonic.

Henceforth, we will require monotonic systems with both separable variables and monotonic increments.

It is easy to see that monotonicity of increments for functions $\pi(i,H)$ with separable variables are uniquely defined by the monotonicity of increments of the associated function $r(H)$ the following must be valid:

$$r(H\cup\ell)-r(H)\le r(E\cup\ell)-r(E) \ \ \forall\, E,H\subseteq W,\ E\subseteq H,\ \ell\notin H\,. \tag{17}$$

3. Local Transformations of a Monotonic System

The means of calculating function $\pi(i,H)$ depends significantly both on the type and form of initial data representation, and on the actual formulation of the structuring problem. Thus, isolation of a class of allowable changes to initial data for solution of the problem of correcting the results of structuring must generally occur in the framework of the characteristic properties of a monotonic system that do not depend on the form of data representation. In other words, we must define the class of allowable changes to initial data in terms of the transformations of the monotonic system itself.

We fix some element $\ell\in W$.

Definition. A positive local transformation of a monotonic system $\langle W,\pi\rangle$ occurs when it is replaced by a new system $\langle W,\pi'\rangle$ for which the following relations are valid:

$$\pi'(i,H)=\pi(i,H)\ \ \forall\, i\in H\ \ (i\neq \ell),\ \forall\, H\subseteq W,\ (\ell\in H), \tag{18}$$

$$\pi'(\ell,H)\geq \pi(\ell,H)\ \ \forall\, H\subseteq W,\ (\ell\in H), \tag{19}$$

$$\pi'(i,H)\leq \pi(i,H)\ \ \forall\, i\in H,\ i\neq \ell,\ \forall\, H\subseteq W,\ (\ell\in H), \tag{20}$$

for any (yet fixed) element ℓ, $\ell\in W$; there exists a set $H\subseteq W$, $\ell\in H$ such that in (19) a strict inequality is satisfied.

A local transformation of a monotonic system will be called negative if, besides (18), the following are satisfied:

$$\pi'(\ell,H)\leq \pi(\ell,H)\ \ \forall\, H\subseteq W,\ (\ell\in H), \tag{21}$$

$$\pi'(i,H)\geq \pi(i,H)\ \ \forall\, i\in H,\ i\neq \ell,\ \forall\, H\subseteq W,\ (\ell\in H), \tag{22}$$

and there exists a set $H\subseteq W$, $\ell\in H$ such that strict inequality is satisfied in (21).

The element ℓ is thus viewed as the fundamental parameter of such transformation.

From (18) and (14) it immediately follows that local transformation of a system with separable variables is determined by the relation

$$\begin{aligned} &p'(i)=p(i)\,\forall\, i\in W, i\neq l,\\ &r'(H)=r(H)\,\forall H\subseteq W, \ell\notin H, \end{aligned} \tag{23}$$

and it is positive, if

$$\begin{aligned} &p'(\ell)>p(\ell),\\ &r'(H)\leq r(H)\,\forall H\subseteq W, \ell\in H, \end{aligned} \tag{24}$$

and negative, if

$$\begin{aligned} &p'(\ell)<p(\ell),\\ &r'(H)\geq r(H)\,\forall H\subseteq W, \ell\in H. \end{aligned} \tag{25}$$

Thus, the simplest local transformation of a monotonic system with separable variables consists in changing one single value of function $p(i)$ while not changing the function $r(H)$, specifically, $p'(\ell)\neq p(\ell)$.

We note that in distribution from local transformation of general monotonic systems, positive and negative transformations of systems with separable variables are characterized by the fact that in inequality (19) or (21) the strict inequality is valid, i.e., for every H, $H \subseteq W$, $\ell \in W$ we have

$$\pi'(\ell,H) > \pi(\ell,H) \tag{26}$$

for positive and

$$\pi'(\ell,H) < \pi(\ell,H) \tag{27}$$

for negative transformations.

We now examine the question of how the defining sequence is changed as a result of a local transformation.

LEMMA 1. *The mutual order of all elements in set W except for the ℓ-th does not change in the defining sequence of a monotonic system with separable variables as a result of local transformation.*

We denote by λ the number of element ℓ in the sequence $I = \langle i_1,...,i_N \rangle$ (i.e., before transformation $\ell = i_\lambda$), and by μ, the number of this same element in sequence $I' = \langle j_1,..., j_N \rangle$ (i.e., after the transformation $\ell = j_\mu$).

LEMMA 2. *As a result of the positive local transformation of a monotonic system with separable variables, element ℓ may be shifted only to the right; $\mu \geq \lambda$, and as a result of a negative transformation, only to the left: $\mu \leq \lambda$.*

By a shift of element ℓ we understand a change in its place (number) in the defining sequence I' of the transformed system $\langle W, \pi' \rangle$ in comparison with its place in sequence I before transformation.

Thus, the following is valid.

Theorem 4. *For conditions of local transformation of a monotonic system with separable variables, the defining sequence* $I' = \langle j_1,..., j_N \rangle$ *of the system* $\langle W, \pi' \rangle$ *is different from the defining sequence* $I = \langle i_1,...,i_N \rangle$ *of the system* $\langle W, \pi \rangle$ *only in the placement of element* ℓ, *and specifically: for a positive transformation we have*

$$\begin{aligned} & j_k = i_k, \quad k = \overline{1,\lambda - 1}, \\ & j_k = i_{k+1}, \quad k = \overline{\lambda, \mu - 1}, \\ & j_\mu = i_\lambda = \ell, \\ & j_k = i_k, \quad k = \overline{\mu + 1, N}. \end{aligned} \tag{28}$$

and for a negative one, respectively

$$\begin{aligned} & j_k = i_k, \quad k = \overline{1,\mu - 1}, \\ & j_\mu = i_\lambda = \ell, \\ & j_k = i_{k-1}, \quad k = \overline{\mu + 1, \lambda}, \\ & j_k = i_k, \quad k = \overline{\lambda + 1, N}. \end{aligned} \tag{29}$$

We note that the placement of element ℓ in defining sequence I for a monotonic system with separable variables after local transformation, i.e., the number μ is determined relatively simply. Actually, if now the inequality

$$p'(\ell) > p(i)$$

is not valid for all elements $i \in W \setminus \ell$, then the number μ is determined from the relation

$$p(i_\mu) \le p'(\ell) \le p(i_{\mu+1}), \tag{30}$$

in the opposite case, $\mu = N$ and conversely, if this inequality is not satisfied for each element $i \in W$, then $\mu = 1$. (In case of nonuniqueness, i.e., when two such elements s and t are encountered that $p'(s) = p'(t)$, the rule for constructing a fixed sequence retains its validity.)

4. Conditions for Correcting the Kernel of a Monotonic System

Formulation and solution are presented in this section for the problem of finding corrections to initial data that are necessary for changing the contents of the kernel of a monotonic system with separable variables and monotonic increments, if the specified set G', which must be obtained as kernel, is different from the obtained set (kernel) G exactly in one element, i.e., $G' = G \cup \ell$ (or $G' = G \setminus \ell$), and ℓ is fixed, and the class of allowable transformations of initial data is limited by the changes in the values of indicators for only one and specifically this element ℓ (i.e., by local transformations).

We examine the problem of local expansion of the kernel of a monotonic system. In other words, we must find a local transformation of an initial monotonic system $\langle W,\pi \rangle$ into $\langle W,\pi' \rangle$ that its kernel G' includes element ℓ, $\ell \in W \setminus G$, and all elements of kernel G of system $\langle W,\pi \rangle$, and only them.

It is easy to see (see Lemma 2) that solution of this problem requires positive local transformations.

Theorem 5. *In order to obtain the set* $G' = G \cup \ell$, *where* $\ell \in W \setminus G$, *as the kernel of monotonic system* $\langle W,\pi' \rangle$, *using positive local transformation of the initial monotonic system* $\langle W,\pi \rangle$ *having separable variables and monotonic increments, it is necessary and sufficient that one of the following three groups of conditions are satisfied*:

I. $$p(i_{m-1}) < p'(\ell) < p(i_m), \tag{31}$$

$$\pi'(\ell, G \cup \ell) \geq \pi(g, G), \tag{32}$$

$$\pi'(j_k, H'_k) < \pi'(\ell, G \cup \ell),\ k = \overline{\chi, m-1}; \tag{33}$$

II. $$p'(\ell) = p(i_m); \tag{34}$$

III. $$p'(\ell) > p(i_m), \tag{35}$$

$$\pi'(j_k, H'_k) \leq \pi'(g, G \cup \ell),\ k = \overline{m, \mu}, \tag{36}$$

where $g = i_m$ is the first element of kernel G in the defining sequence I ; χ is the number of the first element in the quasikernel [6] $\varGamma_{p-1}$*, which directly precedes kernel $G = \varGamma_p$, and elements j_k, $k = \overline{1,N}$, of sequence I' and $\overline{\overline{H}}'$, and set H'_k are defined by the relations of (28), Theorem 4, and relations (7).*

We now examine the opposite problem of local compression of the kernel of a monotonic system. We must find a local transformation (in this case, apparently, negatively) of an initial monotonic system from $\langle W,\pi \rangle$ to $\langle W,\pi' \rangle$, so that its kernel G' does not include element ℓ, $\ell \in G$, yet includes all remaining elements of kernel G and only them.

Theorem 6. In order to obtain, using a negative local transformation, the set $G' = G \setminus \ell$, where $\ell \in G$, as a kernel for a monotonic system with separable variables and monotonic increments, it is necessary and sufficient that the following conditions are satisfied:

$$p'(\ell) < p(i_m), \tag{37}$$

$$\pi'(j_k, H_k) \le F(G \setminus \ell),\ k = \overline{\lambda + 1, N}, \tag{38}$$

$$\pi'(j_k, H'_k) < F(G \setminus \ell),\ k = \overline{1, m-1}, \tag{39}$$

where λ is the number of element ℓ in sequence I before transformation ($\ell = i_\lambda$), and elements j_k, $k = \overline{1,N}$, of sequence I' and set H'_k of sequence $\overline{\overline{H}}$ are determined by relations (29) and (7).

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[6] Quasikernel $\varGamma_{p-1}$ is defined by the relation $F(\varGamma_{p-1}) = \underline{\max}_{k=1,m-1} F(H_k)$ (compare with relations' (8), (9) for kernel $G = \varGamma_p$). The form of notation $(\varGamma_s)$ for quasikernels is in this work unimportant, yet preserved for the sake of succession to works [1, 4-6].

APPENDIX

Proof of Theorems 1-4 (and Lemma 1 and 2). These may be easily obtained sequentially directly from the definition of a p monotonic system, systems with separable variables, positive and negative local transformations, and rule (6) for constructing a defining sequence.

Below we prove Theorem 5. Proof of Theorem 6 is performed analogously and is omitted. A number of lemmas are prerequisite to the proof of Theorem 5.

As is known [1,4], for kernel G of a monotonic system, we have

$$F(H) < F(G) \quad \forall H \supset G, \tag{A.1}$$

$$F(H) \le F(G) \quad \forall H \subseteq G. \tag{A.2}$$

These relations actually are another expression of the definition of a kernel. Thus, in order for set $G \cup \ell$ to be a kernel of system $\langle W, \pi' \rangle$, it is necessary that analogous relations are satisfied for this set.

Relaying on definitions (A.1) and (A.2) and using the property of monotonicity of (1) and the definitions from Sections 2 and 3, we obtain the following assertions.

LEMMA LA1. *The following relation is valid:*

$$\pi(k, G \cup k) < F(G) \quad \forall k \in W \setminus G. \tag{A.3}$$

LEMMA LA2. *For a p-monotonic system, we have*

$$(k, j) \in P, \ (j, k) \notin P \quad \forall j \in G, \ \forall k \in W \setminus G. \tag{A.4}$$

For a system with separable variables, we have

$$p(k) \le p(i_{m-1}) < p(i_m) \le p(j) \quad \forall j \in G, \ \forall k \in W \setminus G, \tag{A.5}$$

where m is the number of the first element in kernel G in the defining sequence I.

LEMMA LA3. *In order for $G' = G \cup \ell$ to be a kernel of a system $\langle W, \pi' \rangle$, it is necessary that the following be true:*

$$p(i_{m-1}) < p'(\ell), \tag{A.6}$$

$$\pi'(\ell, G \cup \ell) \ge \pi(g, G). \tag{A.7}$$

LEMMA LA4. *If we have*

$$p'(\ell) \geq p(i_m),$$

then the condition (32) of the Theorem 5 is fulfilled at the same time.

We note that in proving these assertions, we do not use property (16) regarding monotonicity of increases of function $\pi(i,H)$. Thus, the assertions of lemmas LA1 and LA3 (A.7) are valid for any monotonic system, and lemmas LA3 (A.6) and LA4, for any systems having separable variables.

We also note that, in accordance with Theorem 4 for conditions of positive local transformations of the sequences, the H' and H, defined using (7) for sequences I' and I, respectively, are related as follows:

$$\begin{aligned} &H'_k = H_k, && k = \overline{1,\lambda-1}, \\ &H'_k = H_{k+1} \cup \ell, && k = \overline{\lambda,\mu}, \\ &H'_k = H_k, && k = \overline{\mu+1,N}. \end{aligned} \tag{A.8}$$

Proof of Theorem 5. Necessity. From lemma LA3 (A.5) it follows that if set $G' = G \cup \ell$ is the kernel of monotonic system $\langle W,\pi' \rangle$, then one of the three relations (31), (34), or (35) is valid. We will prove that if (31) is true, then (33) is too. Inequality (32) is satisfied in accordance with the assertion of lemma LA3, and if (35) is true, then so is (36). The necessity of (34) with exclusion of (31) and (35) is evident. At the same time, the requirement that one of three groups of conditions – I, II, or III – will be proved.

Let

$$p(i_{m-1}) < p'(\ell) < p(i_m) = p(g).$$

We assume that inequality (33) is violated, i.e., there exists a number k, $k = \overline{\chi, m-1}$, such that

$$\pi'(j_k, H'_k) \geq \pi'(\ell, G \cup \ell) \geq F'(G \cup \ell)$$

($\chi = i \in \Gamma_{p-1}$, is the number of the first element of the quasikernel Γ_{p-1}, see the footnote Theorem 5).

This inequality contradicts relation (8) for set $G \cup \ell$ as a kernel. At the same time, the necessity of (33) for the case where the set $G' = G \cup \ell$ becomes a kernel is proved. [7]

Now let

$$p'(\ell) > p(i_m).$$

We assume that inequality (36) is violated, i.e., there exists a number k, $k = \overline{m,\mu}$, such that

$$\pi'(j_k, H'_k) > \pi'(g, G \cup \ell) \geq F'(G \cup \ell).$$

This inequality contradicts (9) for the set $G \cup \ell$. Necessity is proved. [8]

<u>Sufficiency</u>. Next we prove the sufficiency of satisfying each of the three groups of conditions.

<u>Group I</u>. In this case, in relation to (A.8) and a comment regarding the definition of number μ, evidently, we have

$$G \cup \ell \in \overline{H}'$$

and

$$F'(G \cup \ell) = \pi'(\ell, G \cup \ell).$$

Since (31) denotes $\mu = m - 1$, then taking account of the definition of a positive local transformation from (33) and the definition of a quasikernel Γ_{p-1}, it follows that

$$\pi'(j_k, H'_k) < \pi'(\ell, G \cup \ell) = F'(G \cup \ell),\ k = \overline{1, m-2},$$

which corresponds to relation (8) for the set $G \cup \ell$.

Relation (9) follows from $\mu = m - 1$, $H'_{m-1} = G \cup \ell$, and respectively, $H'_m = G$, i.e., [9]

$$\pi'(j_k, H'_k) = \pi(i_k, H_k) \leq \pi(g, G) \leq \pi'(\ell, G \cup \ell)\ \ \forall H_k \subseteq G.$$

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[7] Thus, the necessity of (33) is proved independently of (31), (34), or (35). However, as shown below (for proof of sufficiency), satisfaction of (34) or (35) comes forth as an added necessity condition.

[8] Thus, the necessity of one of three groups of conditions in Theorem 5 is proved without using property (16) for monotonicity of increments in the function $\pi(i, H)$, i.e., this part of Theorem's assertion is valid for any monotonic system with separable variables.

[9] We note that in order for set $G' = G \cup \ell$ of system $\langle W, \pi' \rangle$ after a positive local transformation of an arbitrary monotonic system $\langle W, \pi \rangle$ with separable variables, satisfaction of group I conditions of the Theorem 5 is sufficient.

Group II. Since $p'(\ell)=p(i_m)>p(i_{m-1})$, then in this case we also have

$$G\cup\ell\in\overline{H}',$$

and $\mu\geq m-1$, and

$$F'(G\cup\ell)=\pi'(\ell,G\cup\ell)=\pi'(g,G\cup\ell).$$

We examine the set H'_k, $k=\overline{\lambda,\mu-1}$ of sequence $\overline{H}'$. In accordance with (A.8) they are related to sets H_k, $k=\overline{\lambda,\mu-1}$ of the sequence $\overline{H}$ as follows:

$$H'_k=H_{k+1}\cup\ell,\ k=\overline{\lambda,\mu-1}.$$

From the definition of a monotonic system with separable variables (14), (15) and monotonic increments (16), (17) for any set H'_k, $k=\overline{\lambda,m-2}$ $(H'_k\supset G\cup\ell)$, we have

$$r'(H_{k+1}\cup\ell)-r(H_{k+1})\leq r'(G\cup\ell)-r(G),$$

from which, taking account of

$$F(H_{k+1})=\min_{i\in H_{k+1}}\pi(i,H_{k+1})<\pi(g,G)=F(G),\ k=\overline{\lambda,m-2},$$

it follows that

$$F'(H'_k)=\min_{i\in H'_k}\pi'(i,H'_k)<\pi'(g,G\cup\ell)=F'(G\cup\ell),\ k=\overline{\lambda,m-2} \quad \text{(A.9)}$$

We now examine the set H'_k of the sequence $\overline{H}'$ when $k=\overline{1,\lambda-1}$ and $k=\overline{m,N}$. In accordance with (A.8) they are related to sets H_k by sequence $\overline{H}$ as follows:

$$H'_k=H_k,\ k=\overline{1,\lambda-1},\ k=\overline{m,N}.$$

Then, from the definition of a positive local transformation for these sets ($\ell\in H_k$, $k=\overline{1,\lambda-1}$ and $\ell\in H_k$, $k=\overline{m,N}$) we have

$$F'(H'_k)=F'(H_k)\leq F(H_k)<F(G)\leq F'(G\cup\ell),\ k=\overline{1,\lambda-1} \quad \text{(A.10)}$$

$$F'(H'_k)=F'(H_k)=F(H_k)\leq F(G)\leq F'(G\cup\ell),\ k=\overline{m,N}. \quad \text{(A.11)}$$

Thus, satisfaction of (34) from Theorem 5 also entails the validity of both relations (8) and (9) for set $G\cup\ell$ (the first of them is obtained by joining (A.9) and (A.10)).

Group III. Condition (35), Theorem 4 and (A.8) denote that $G \cup \ell \in \overline{H}'$, so that

$$F'(G \cup \ell) = \pi'(g, G \cup \ell) < \pi'(\ell, G \cup \ell).$$

For set H'_k, $k = \overline{1, m-2}$ the questions raised earlier for group II condition are, in this case, fully valid, and consequently, relation (8) (which determines the kernel) is valid.

Analogously for sets H'_k, $k = \overline{\mu+1, N}$ in this case a relation like (A.11) is valid.

Combining it with condition (36), we obtain the second relation in (9) for determining the kernel for the set $G \cup \ell$. The theorem is proved. ■

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