Linguistic Analysis of 0-1 Matrices using Monotone Systems *

I. B. Muchnik, N. F. Chkuaseli and L. V. Shvartser UDC 512.932:65.012.122

We examine a linguistic method of the analysis of 0-1 matrices in which a matrix is approximated by a small number of submatrices. An efficient algorithm using the apparatus of monotone system is proposed for optimizing the partition.**

1. Introduction

Currently available linguistic methods for data matrix analysis are designed for the processing of numerical "object-attribute" matrices [1,2]. In practice, especially in social sciences, economics, and medical research, empirical information is often collected as qualitative data. Such information is always easily and as shown in [3], expediently presented in the form of 0-1 matrices. Typical examples of 0-1 matrices are provided by studies of organizational structure and operation [3]. In this article, we perform linguistic analysis of "object-attribute" matrices with 0-1 elements.

The proposed method partitions the "object-attribute" matrix into vertical "contrasts" bands, and the intuitive notion of contrast is formalized with the aid of monotone systems as in [3]. As a result, the attributes are partitioned into a given number of groups, and in each group the objects are separated into two classes, which consist "almost of 1-s" and "almost of 0-s" respectively. For 0-1 matrices, this special partition of objects is of considerable interest, because the outcome has an appealingly simple interpretation. In the analysis of numerical data, on the other hand, many classes are usually needed in order to classify the objects within the attribute groups [1].

The algorithms are computationally efficient because the particular family of monotone functions used allows the functionals being extremized (the functional increments) to be efficiently updated following a local change in the vertical bands(removal or insertion of one column).

---

* Moscow, Tbilisi. Translated from Avtomatica i Telemekhanika, No. 4, pp. 132 – 139, April, 1986. Original article submitted November 13, 1984.

** Only one explicit reference to Monotone Systems [4] in the literature list cited at the end of this article was found in Appendix. In [4] the "Monotone System" referred is called "Monotonic System", noted by JM.

2. The problem of partition of a 0-1 matrix into contrast bands using monotone systems

Let W be the set of objects ($|W| = N$) and Y the set of attributes ($|Y| = M$), Φ is the "object-attribute" matrix with 0-1 elements,

$$\Phi = \|\varphi_{i\,j}\|,\ \varphi_{i\,j} = \begin{cases} 1 & \text{when attribute } j \text{ corresponds to object } i \\ 0 & \text{otherwise} \end{cases}.$$

Consider the partition of the set Y into k nonintersecting subsets (classes) ($Y = \bigcup_{q=1}^{k} Y^q$, $Y^{q_1} \cap Y^{q_2} = \varnothing$, $q_1 \neq q_2$). The matrix Φ is thus partitioned into k nonintersecting submatrices $\Phi = (\Phi^1, \Phi^2, \ldots, \Phi^k)$, such that the q-th submatrix is a correspondence of the set W and some Y^q.

For each q, we define the monotone system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ by [1]

$$\pi_\alpha^q(i, H) = \alpha \cdot |Y_H^q| - (1-\alpha) \cdot |y_i^q|, \tag{1}$$

where $H \subseteq W$, $i \in H$; y_i^q is the set of attributes corresponding to the object i in the q-th class of attributes; $Y_H^q = \bigcup_{j \in H} y_j^q$; α is a numerical parameter, $0 \le \alpha \le 1$. The system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ is called a particular monotone system.

The value of $|y_i^q|$ may be interpreted as the "degree of shading" of the row i in the band Φ^q (we assume that the cell (i, j) is "shaded" if $\varphi_{i\,j} = 1$) The value of $|Y_H^q|$ may be interpreted as some measure of overall "shading" of the set H in the band Φ^q.

Then $\pi_\alpha^q(i, H)$ in some sense characterizes the deviation of individual "shading" from the group "shading" in the band Φ^q, with certain weights assigned to individual and group "shadings".

The monotone system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ generates on the set 2^W of all subsets of the set W the characteristic function

$$F_\alpha^q(H) = \min_{i \in H} \pi_\alpha^q(i, H),\ H \subseteq W.$$

[1] For $\alpha = ½$, the monotone system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ is the system $\langle W, \pi_2 \rangle$ [3].

The maximum-cardinality set $G_\alpha^q \subseteq W$, on which the function $F_\alpha^q(H)$ attains its global maximum, is called the nucleus of the monotone system $\langle W, \pi_\alpha^q, \Phi^q \rangle$.

The monotone system (1) partitions the set W into G_α^q and $W \setminus G_\alpha^q$. The next proposition shows that the "degree of shading" of the elements in the nucleus is always less than the "degree of shading" of the elements in its complement.

Theorem 1. Let $i_1 \in G_\alpha^q$, $i_2 \in W \setminus G_\alpha^q$. Then for all α $(0 \le \alpha \le 1)$ we have $\left|y_{i_1}^q\right| < \left|y_{i_2}^q\right|$.

The proof is given in the Appendix.

For small values of α the set G_α^q is lightly "shaded" (most of the $\left|y_i^q\right|$ are small). Conversely, the set $W \setminus G_\alpha^q$ consists of heavily "shaded" elements. Small values of $\pi_\alpha^q(i,H)$ on $W \setminus G_\alpha^q$ guarantee a low scatter of the sets y_i^q in $Y_W^q \setminus G_\alpha^q$, i.e., homogeneity of the set $W \setminus G_\alpha^q$ in the band Φ^q. As the parameter α is increased $\left|Y_H^q\right|$ becomes larger and the "shading" contrast between the sets G_α^q and $W \setminus G_\alpha^q$ is reduced. [2]

Thus, the "contrast" of the bands Φ^q is naturally measured by $F_\alpha^q(G_\alpha^q)$. The problem of identifying "contrast" bands in the matrix Φ thus may be stated as follows. It is required to find a partition of the attribute set Y into k nonintersecting subsets $Y^1, \dots, Y^k$ (respectively a partition of the matrix Φ into k bands $\Phi^1, \dots, \Phi^k$ so as to maximize the functional

$$J_\alpha^1(R) = \sum_{q=1}^{k} J_\alpha^q , \tag{2}$$

where

$$J_\alpha^q = F_\alpha^q(G_\alpha^q), \tag{3}$$

on the set of all partitions $R = \left\{Y^1, \dots, Y^k\right\}$ of Y.

Other measures of band contrasts may be introduced. Thus, for instance, it is easily seen that the measure (1) only accounts for the "degree of shading" of the nucleus G_α^q and

---

[2] For large α ($\alpha \ge ½$), the sets G_α^q and $W \setminus G_\alpha^q$ also have a high contrast, although in a certain different sense. In this study, we will only consider "shading" contrast.

ignores the “shading” differences between the sets G_α^q and $W \setminus G_\alpha^q$. We can therefore modify (3) as follows:

$$\left(J_\alpha^q\right)^{\text{mod}} = F_\alpha^q(G_\alpha^q) - F_\alpha^q(W). \tag{4}$$

The functional being maximized changes accordingly:

$$J_\alpha^2(R) = \sum_{q=1}^{k} \left(J_\alpha^q\right)^{\text{mod}}. \tag{5}$$

To optimize either functional (2) or (5), we can follow a standard general scheme. [3]

Starting with an arbitrary partition $R = \{Y^1, Y^2, ..., Y^k\}$, we sequentially try to remove from each class Y^i, $i = \overline{1,k}$ the attribute j, $j \in Y^i$ and at the same time try to insert it sequentially into each class t ($t = \overline{1,k}$, $t \neq i$) of the partition R.

In the process we compute the value of (3), J_α^{i-j} on $Y^i \setminus \{j\}$ and J_α^{t+j} on $Y^t \cup \{j\}$. The element j is inserted in the class t on which the functional (2) has the largest increment. This increment is given by the formula

$$\Delta_{i\,t}^{j} J_\alpha(R) = J_\alpha^{i-j} - J_\alpha^{i} + J_\alpha^{t+j} - J_\alpha^{t}. \tag{6}$$

A compute trial-and-error cycle of placing all the elements in all the classes and then relocating the elements in accordance with the maximum increment of the functional (2) will generate the partition R^1 of Y.

If this partition is different from R, i.e., if at least one $\Delta_{i\,t}^{j} J_\alpha(R)$ is strictly positive, we restart the algorithm with $R = R^1$. Otherwise, the algorithm ends. Partition R is the sought partition.; the sought vertical bands are correspondingly the submatrices $\Phi^1, \Phi^2, ..., \Phi^k$, partitioned into horizontal bands $\left\langle \left(G^1, W \setminus G^1\right), ..., \left(G^k, W \setminus G^k\right) \right\rangle$.

We see from the proposed algorithm that the main computational operation is the evaluation of $\Delta_{i\,t}^{j} J_\alpha(R)$. Efficient computation of this increment requires exploiting the specific features of the monotone system used.

---

[3] All the algorithms are described only for (2), since for (5) the algorithms are entirely similar. The evaluation of (5) is no more complex than the evaluation of (2), since $F_\alpha^q(W)$ are computes by the procedure used to identify the nuclei of the system $\left\langle W, \pi_\alpha^q, \Phi^q \right\rangle$.

3. Algorithm to identify the nucleus of a particular monotone system $\langle W, \pi_\alpha^q, \Phi^q \rangle$

The algorithm makes use of the following fact.

Theorem 2. Let the order P^q be defined by the sequence of indexes in the series

$$\left|y_{i_1^q}^q\right| \geq \left|y_{i_2^q}^q\right| \geq \ldots \geq \left|y_{i_N^q}^q\right|.$$

Then the nucleus of the particular monotone system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ is one of the sets H_ℓ^q, $\ell = \overline{1, N}$, where $H_\ell^q = \{i_\ell^q, i_{\ell+1}^q, \ldots, i_N^q\}$.

The proof is given in the Appendix.

We use the notation: $I^{pq} = \langle i_1^q, i_2^q, \ldots, i_N^q \rangle$.

The algorithm to find the nucleus of the system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ thus involves constructing the sequence I^{pq}, computing N values $\pi_\alpha^q(i_\ell^q, H_\ell^q)$, and identifying as G_α^q the first of the sets H_ℓ^q on which $F_\alpha^q(H_\ell) = \pi_\alpha^q(i_\ell^q, H_\ell^q)$, $\ell = \overline{1, N}$, is maximized.

Updating the Nucleus Inside the Band after a Local Change of the Band. The increment of the functional (2) can be determined directly from (6) by a two-fold application of the algorithm that computes the nucleus of $\langle W, \pi_\alpha^q, \Phi^q \rangle$. Yet it is often essentially easier to update J_α^{q-j} or J_α^{q+j} for known I^{pq}, G_α^q, J_α^q than to perform a complete reordering and to compute N values of $\pi_\alpha^q(i, H)$.

The elements $i_1, i_2 \in W$ are called p^q-equal if $\left|y_{i_1}^q\right| = \left|y_{i_2}^q\right|$. Clearly, p^q-equal elements may be placed in an arbitrary order in the sequence I^{pq}. The number of p^q-equal elements largely determines the complexity of the nucleus-updating algorithm inside some band Φ^q.

Let I^{pq}, G_α^q, J_α^q, respectively, be the defining sequence, [4] the nucleus, and the value of (3) for the system $\langle W, \pi_\alpha^q, \Phi^q \rangle$.

Without loss of generality, we may assume that every pair of p^{q+j}-equal (p^{q-j}-equal) elements preserves their order in I^{pq}.

[4] Concept of "defining sequence" was first introduced in 1971, see http://www.datalaundering.com/download/modular.pdf, or [4], noticed by JM.

Theorem 3. When the band Φ^q is changed precisely by one column j, the order I^{pq} is transformed into order I^{pq+j} (I^{pq-j}), which coincided with the order I^{pq} on the set of elements that are not p^q-equal.

This proposition follows in an obvious way from the fact that when the band Φ^q is changed by one column, for all $i \in W$ the $\left|y_i^q\right|$ either remains unchanged or change precisely by 1 (increasing when a column is inserted and decreasing when a column is removed).

Theorem 4. The sets H_ℓ^q and H_ℓ^{q+j} (H_ℓ^{q-j}) coincide if the element ℓ of W has the same place in the order I^{pq} and in the order I^{pq+j} (I^{pq-j}).

The proposition follows from the fact that the elements of the defining sequence are shifted only in one direction: to the left when a column is added and to the right when a column is removed.

Theorem 4 leads to simple formulas for updating $F_\alpha^{q-j}(H_\ell)$ ($F_\alpha^{q+j}(H_\ell)$) in those cases when the element ℓ retains its position in the defining sequence as a result of a one-column change in the band Φ^q:

$$\pi_\alpha^{q-j}(i_\ell, H_\ell) = \pi_\alpha^{q-j}(i_\ell, H_\ell) - \alpha \cdot \max_{i \in H_\ell} \varphi_{ij} + (1-\alpha) \cdot \varphi_{ij}, \tag{7}$$

$$\pi_\alpha^{q+j}(i_\ell, H_\ell) = \pi_\alpha^{q+j}(i_\ell, H_\ell) + \alpha \cdot \max_{i \in H_\ell} \varphi_{ij} - (1-\alpha) \cdot \varphi_{ij}. \tag{8}$$

Formulas for complete re-computation do not involve computation of $\left|Y_{H_\ell}^q\right|$ and therefore formulas (7) and (8) are essentially simpler than of (1). Since the place of an element in the sequence is not preserved only for p^q-equal elements, substantial computational savings may be achieved if there are only few such elements.

The step updating the nucleus G_α^{q+j} (G_α^{q-j}), the defining sequence I^{pq+j} (I^{pq-j}), and the functionals J^{q+j} (J^{q-j}) are thus designed in the following way. Suppose that the previous step generated the characteristics I^{pq}, G_α^q, J_α^q, $\left|y_i^q\right|$, $i = \overline{1,N}$, $\pi_\alpha^q(i_\ell^q, H_\ell^q)$, and also identified the classes of p^q-equal elements. It is required to insert [5] the j-th column in the matrix Φ^q and to update all the characteristics.

[5] For the removal of the j-th column the algorithm is obviously the same.

1) Construct the sequence G_{α}^{q+j}. Each of the classes Q of p^q-equal elements is partitioned into two subsets Q_1 and the elements of Q_2, where

$$Q_1 = \left\{ i \,\middle|\, i \in Q, \varphi_{i\,j} = 1 \right\},\; Q_2 = \left\{ i \,\middle|\, i \in Q, \varphi_{i\,j} = 0 \right\}$$

The elements of these classes are renumbered so that the sequence I^{pq-j} starts with the elements of Q_1 and the elements of Q_2 follow.

2) Update $\pi_{\alpha}^{q+j}(i_{\ell}, H_{\ell})$. Compare the indexes of the elements in the set W and in the sequences I^{pq} and I^{pq+j}. If they coincide, then use formula (7); otherwise, use formula (1). In conclusion of this step, find the nucleus G_{α}^{q+j} and use (3) to compare J.

Remark 1. Higher computational efficiency may be achieved in computing the increment (6). Thus, using criterion (2) with $\alpha = ½$ (for this value of α, each particular monotone system coincides with the monotone system $\langle W, \pi_2 \rangle$ of [3] on the corresponding submatrix), we can improve the computational efficiency in the following way.

Examine Y^i and Y^t. Let m_i be the first element of the nucleus $G_{½}^{i}$ in the sequence I^{pi}, m_t the first element of the nucleus $G_{½}^{t}$ in the sequence I^{pt} and $j \in Y^i$.

Theorem 5. $\Delta_{i\,t}^{j} J_{½}(R) > 0$ if and only if the following conditions hold:

1) there exists a set H_{ℓ}^{i}, $\ell \geq m_i$, such that

$$F_{½}^{i}(H_{\ell}^{i}) = F_{½}^{i}(G_{½}^{i}) \text{ and } F_{½}^{i-j}(H_{\ell}^{i}) = F_{½}^{i}(G_{½}^{i});$$

2) there exists a set H_{ℓ}^{t}, $\ell \geq m_t$, such that

$$F_{½}^{t}(H_{\ell}^{t}) = F_{½}^{t}(G_{\ell}^{t}) \text{ and } F_{½}^{t+j}(H_{\ell}^{t}) = F_{½}^{t}(G_{\ell}^{t}) + 1/2\,.$$

Thus, when computing (6) in this case, we only have to enumerate the subsets “after” (inside) the nucleus, which satisfy the conditions 1) and 2).

Remark 2. Each vertical band may be partitioned into more than two classes with different degrees of shading. One such possibility is provided by the nested system of subsets $W = (\Gamma_1)_{\alpha}^{q} \supset (\Gamma_2)_{\alpha}^{q} \supset \ldots \supset (\Gamma_p)_{\alpha}^{q} = G_{\alpha}^{q}$, such that

$$F_{\alpha}^{q}((\Gamma_j)_{\alpha}^{q} > F_{\alpha}^{q}(H),\; W \supseteq H \supset (\Gamma_j)_{\alpha}^{q},\; j = \overline{1,p}\,.$$

In this case, we may maximize the criterion (2) or (5), but the vertical bands obtained in the process should be partitioned into p (and not 2) classes $\left(\Gamma_p\right)_\alpha^q, \left(\Gamma_{p-1}\right)_\alpha^q \setminus \left(\Gamma_p\right)_\alpha^q, \ldots, \left(\Gamma_1\right)_\alpha^q \setminus \left(\Gamma_2\right)_\alpha^q$ (p may differ for different vertical bands). Another independent statement of this problem may be obtained using the following construction. Let $\left(G_\alpha^q\right)^0; \left(G_\alpha^q\right)^1; \ldots; \left(G_\alpha^q\right)^s$ be the sequence of nuclei of the monotone systems $\left\langle W, \pi_\alpha^q, \Phi^q \right\rangle; \left\langle W \setminus \left(G_\alpha^q\right)^0, \pi_\alpha^q, \Phi^q \right\rangle; \ldots \left\langle W \setminus \left(\left(G_\alpha^q\right)^0 \cup \left(G_\alpha^q\right)^1 \cup \ldots \cup \left(G_\alpha^q\right)^s\right), \pi_\alpha^q, \Phi^q \right\rangle$, respectively. Then maximizing a criterion of the form (2), where

$$J_\alpha^q = \sum_{n=0}^{s} F_\alpha^q\left(\left(G_\alpha^q\right)^n\right)$$

(s is different for different vertical bands) we obtain a partition of the matrix Φ into vertical bands, which are divided into classes with different degrees of shading. Note, however, that this statement of the problem, unlike the first one, is associated with considerable computational difficulties.

4. Examples of identifying contrast bands on "object-attribute" matrices with 0-1 elements

Example 1. Consider the matrix

$$\Phi_1 = \begin{Vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{Vmatrix}$$

Let the set of columns Y of Φ_1 be partitioned into two nonempty sets in some way, e.g.,

$$Y^1 = \{1,3,5,7\},\ Y^2 = \{2,4,6\}.$$

Applying the proposed algorithm to optimize the functional (2) for $0 \le \alpha \le 1/3$, we obtain the column partition

$$Y^1 = \{1,2,3\},\ Y^2 = \{4,5,6,7\}. \tag{9}$$

Each of the bands Φ^1, Φ^2 is split into two classes:

$$\begin{aligned} &G^1 = \{4,5,6,7\}, \quad W \setminus G^1 = \{1,2,3\} \\ &G^2 = \{1,2,3\}, \quad\;\; W \setminus G^2 = \{4,5,6,7\} \end{aligned}. \tag{10}$$

In this range of α, the proposed algorithm in effect diagonalizes the association matrix [2]. It is also easily seen that in this simple case of full blocks of 1-s with complements consisting entirely of 0-s, the proposed algorithm diagonalizes any matrix Φ, regardless of its dimension and the dimension of the blocks.

For $1/3 \le \alpha < 1/2$, the partition (9), (10) is not the only optimal partition: "non-contrast" partitions are also optimal in the sense of $J_\alpha^1(R)$, such as

$$Y^1 = \{1,2,3,4,5,6\},\ Y^2 = \{7\},$$

$$G^1 = W,\ W \setminus G^1 = \varnothing,$$

$$G^2 = \{1,2,3\},\ W \setminus G^2 = \{4,5,6,7\}.$$

Using (5) instead of (2) leads to somewhat better results: the criterion (5) identifies a unique optimal partition (9), (10) in wider range of α than criterion (2) ($0 \le \alpha < 1/2$).

<u>Example 2</u>. Now consider a matrix in which block of 1-s contain zeros and blocks of zeros contain 1-s:

$$\Phi_2 = \left\| \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{matrix} \right\|$$

Using the functional (2) and $0 \le \alpha < 3$, the sets Y^1 and Y^2 as in Example 1 take the form (9), and the sets G^1 and G^2 are

$$G^1 = \{4,7\},\ G^2 = \{1\},$$

i.e., the nuclei of the systems $\langle W, \pi_\alpha^1, \Phi_1 \rangle$ and $\langle W, \pi_\alpha^2, \Phi_1 \rangle$ contain only zero rows. For $1/3 \le \alpha < 1/2$, one of the optimal partitions is (9), (10); however, other partitions are also optimal, such as

$Y^1 = \{1,3,5,7\}$, $\qquad Y^2 = \{2,4,6\}$,

$G^1 = \{1,3,4,5,7\}$, $\qquad W \setminus G^1 = \{2,6\}$,

$G^2 = \{1,2,4\}$, $\qquad W \setminus G^2 = \{3,5,6,7\}$.

The criterion (5) for $1/3 \le \alpha < 1/2$ "identifies" the block-diagonal structure (9), (10) of the matrix Φ_2 and does not identify structures, which are essentially far from block-diagonal. Specifically, the only other optimal partition for the given criterion is a partition, which is "close" to (9), (10),

$Y^1 = \{1,2\}$, $\qquad Y^2 = \{3,4,5,6,7\}$,

$G^1 = \{4,6,7\}$, $\qquad W \setminus G^1 = \{1,2,3,5\}$,

$G^2 = \{1,2,3\}$, $\qquad W \setminus G^2 = \{4,5,6,7\}$.

APPENDIX

<u>Proof of Theorems 1 and 2</u>.

We start with some definitions. Let a non-strict linear order P be defined on a finite set W ($|W| = N$). It orders all elements of the set into the sequence $I^P = \langle i_1^P, i_2^P, .., i_N^P \rangle$, where $(i_k^P, i_t^P) \in P$ for $k \le t$ up to p-equal elements (elements x and y are p-equal if $(x,y) \in P$ and also $(y,x) \in P$).

$\langle W, \pi, P \rangle$ is a p-monotone system if

1) $\langle W, \pi \rangle$ is a monotone system;

2) $\pi(x,H) < \pi(y,H)$ for all $(x,y) \in P$ and $(y,x) \notin P$;

3) $\pi(x,H) = \pi(y,H)$ for equal x and y (for every $H \subseteq W$).

In [3,4], the nucleus of a monotone system is formed by constructing the defining sequence $I = \langle i_1, i_2, ..., i_N \rangle$:

$$\pi(i_\ell, H_\ell) = \min_{i \in H_\ell} \pi(i, H_\ell), \ \ell = \overline{1,N}, \tag{A.1}$$

where $H_\ell = W \setminus \{i_1, ..., i_{\ell-1}\}$, $H_1 = W$, the nucleus $G = H_m = \{i_m, ..., i_N\}$ is defined by

$$\pi(i_\ell, H_\ell) < \pi(i_m, H_m),\ \ell = \overline{1, m-1},$$
$$\pi(i_\ell, H_\ell) \le \pi(i_m, H_m),\ \ell = \overline{m, N}. \qquad \text{(A.2)}$$

LEMMA A.1. The defining sequence of a p-monotone system coincides, up to permutations of p-equal elements, with the sequence I^P.

LEMMA A.2. If the element x is included in the nucleus of a p-monotone system $\langle W, \pi, P \rangle$ ($x \in G$), then all the elements of the set W p-equal to x are also included in the nucleus.

Corollary. The algorithm to identify the nucleus of a p-monotone system reduces to evaluating $\pi(i_\ell, H_\ell)$, $\ell = \overline{1, N}$ on the sequence I^P. The nucleus G is identified with the set H_m satisfying (A.2).

It is easily seen that the complexity of finding G by this algorithm is directly proportional to N, whereas for a general algorithm it is directly proportional to N^2.[6]

Proof of Lemma A.1. Suppose that the defining sequence $I = \langle i_1, i_2, ..., i_N \rangle$ has been constructed. Take two elements i_k and i_t such that $k < t$.

By (A.1) we have

$$\pi(i_k, H_k) \le \pi(i_t, H_t). \qquad \text{(A.3)}$$

From the definition of the order P, there are three possible relations between the elements i_k and i_t: $(i_k, i_t) \in P$, or $(i_t, i_k) \in P$, or finally the elements i_k and i_t are p-equal. But using property (2) of p-monotonicity and the inequality (A.3), we conclude that the case $(i_t, i_k) \in P$ and $(i_k, i_t) \notin P$ is impossible. Therefore,

$$(i_k, i_t) \in P.$$

Thus, for all $k, t = \overline{1, N}$ such that $k < t$ the elements of the defining sequence are in the order relation P. ■

[6] More effective $O(N \cdot log_2 N)$ general procedure for nucleus (kernel) search by constructing non-complete defining sequence may be found at http://www.datalaundering.com/download/classarv.pdf , noticed by JM.

Proof of Lemma A.2. The proof is by contradiction. Let the elements x and y be p-equal, while $x \in G$ and $y \notin G$.

Consider the set $G \cup \{y\}$. By monotonicity of $\pi(i,H)$, for every $i \in G$, we have

$$\pi(i,G \cup \{y\}) \geq \pi(i,G). \tag{A.4}$$

By the third monotonicity property and (A.4),

$$\pi(y,G \cup \{y\} = \pi(x,G \cup \{y\}) \geq \pi(x,G). \tag{A.5}$$

From (A.4) and (A.5) we have

$$F(G \cup \{y\}) = \min_{i \in G \cup \{y\}} \pi(i,G \cup \{y\}) \geq \min_{i \in G} \pi(i,G) = F(G),$$

which contradicts the definition of the nucleus of a monotone system. ■

LEMMA A.3. The system $\langle W, \pi_\alpha^q, \Phi^q \rangle$ is p-monotone, and its order P is defined by the sequence of indexes in the series

$$\left| y_{i_1^q}^q \right| \geq \left| y_{i_2^q}^q \right| \geq \ldots \geq \left| y_{i_N^q}^q \right|.$$

The proof of this lemma reduces to direct verification of the p-monotonicity properties for the system $\langle W, \pi_\alpha^q, \Phi^q \rangle$.

Theorem 1 follows from Lemmas A.2 and A.3, and Theorem 2 follows from lemmas A.1 and A.3.

LITERATURE CITED

1. E. M. Braverman, N. E. Kiseleva, I. B. Muchnik, and S.G. Novikov, "A linguistic approach to processing of large data files," *Avtomat. Telemekh., No. 1*, 73-88 (1974).
2. E. M. Braverman and I. B. Muchnik, Structural Methods of Empirical Data Processing [in Russian], *Nauka*, Moscow (1983).
3. E. N. Kuznetsov and I. B. Muchnik, "Analysis of the allocation of functions in an organization," *Avtomat. Telemekh., No. 19*, 119-127 (1982), http://www.datalaundering.com/download/organiza.pdf .
4. I. E. Mullat, "Extremal subsystems of monotone systems, I, II," *Avtomat. Telemekh., No. 5*, 130-139 (1976), http://www.datalaundering.com/download/extrem01.pdf , *No.8*, 169-179 (1976), http://www.datalaundering.com/download/extrem02.pdf .