

# Duality between quasi-concave functions and monotone linkage functions

Yulia Kempner<sup>1</sup> and Vadim E. Levit<sup>1,2</sup>

<sup>1</sup>Dept. of Computer Science  
Holon Institute of Technology  
yuliak@hit.ac.il

<sup>2</sup>Dept. of Computer Science and Mathematics  
Ariel University Center of Samaria  
levitv@ariel.ac.il

## Abstract

A function  $F$  defined on all subsets of a finite ground set  $E$  is *quasi-concave* if  $F(X \cup Y) \geq \min\{F(X), F(Y)\}$  for all  $X, Y \subset E$ . Quasi-concave functions arise in many fields of mathematics and computer science such as social choice, theory of graph, data mining, clustering and other fields.

The maximization of quasi-concave function takes, in general, exponential time. However, if a quasi-concave function is defined by associated monotone linkage function then it can be optimized by the greedy type algorithm in a polynomial time.

Quasi-concave functions defined as minimum values of monotone linkage functions were considered on antimatroids, where the correspondence between quasi-concave and bottleneck functions was shown [5]. The goal of this paper is to analyze quasi-concave functions on different families of sets and to investigate their relationships with monotone linkage functions.

## 1 Preliminaries

Many combinatorial optimization problems can be formulated as: for a given *set system* over  $E$  (i.e., for a pair  $(E, \mathcal{F})$  where  $\mathcal{F} \subseteq 2^E$  is a family of feasible subsets of finite set  $E$ ), and for a given function  $F : \mathcal{F} \rightarrow \mathbf{R}$ , find an element of  $\mathcal{F}$  for which the value of the function  $F$  is extremal. In general, this optimization problem is **NP**-hard, but for some specific functions and set systems the problem may be solved in polynomial time. For instance, modular cost functions can be optimized over matroids by greedy algorithms [4], and bottleneck functions can be maximized over greedoids [2]. Another example is about set functions defined as minimum values of monotone linkage functions. These functions are known as quasi-concave set functions. Such set functions can be maximized

by a greedy type algorithm over the family of all subsets of  $E$  [6],[10],[14],[16], over antimatroids and convex geometries [5],[7],[11], join-semilattices [13] and meet-semilattices [8].

Originally [9], these functions were defined on the Boolean  $2^E$ :

$$\text{for each } X, Y \subset E, F(X \cup Y) \geq \min\{F(X), F(Y)\}. \quad (1)$$

In this work we extend this definition to set systems that are not necessarily closed under union.

Let  $E$  be a finite set, and a pair  $(E, \mathcal{F})$  be a set system over  $E$ .

**Definition 1.1** *A minimal feasible subset of  $E$  that includes a set  $X$  is called a cover of  $X$ .*

We will denote by  $\mathcal{C}(X)$  the family of covers of  $X$ .

**Definition 1.2** *A function  $F$  defined on a set system  $(E, \mathcal{F})$  is quasi-concave if for each  $X, Y \in \mathcal{F}$ , and  $Z \in \mathcal{C}(X \cup Y)$ ,*

$$F(Z) \geq \min\{F(X), F(Y)\}. \quad (2)$$

If a set system is closed under union, then the family of covers  $\mathcal{C}(X \cup Y)$  contains the unique set  $X \cup Y$ , and the inequality (2) coincides with the original inequality (1).

Here we give definitions of some set properties that are discussed in the following section. We will use  $X \cup x$  for  $X \cup \{x\}$ , and  $X - x$  for  $X - \{x\}$ .

**Definition 1.3** *A non-empty set system  $(E, \mathcal{F})$  is called accessible if for each non-empty  $X \in \mathcal{F}$ , there is an  $x \in X$  such that  $X - x \in \mathcal{F}$ .*

For each non-empty set system  $(E, \mathcal{F})$  accessibility implies that  $\emptyset \in \mathcal{F}$ .

**Definition 1.4** *A closure operator,  $\tau : 2^E \rightarrow 2^E$ , is a map satisfying the closure axioms:*

- C1:  $X \subseteq \tau(X)$*
- C2:  $X \subseteq Y \Rightarrow \tau(X) \subseteq \tau(Y)$*
- C3:  $\tau(\tau(X)) = \tau(X)$ .*

**Definition 1.5** *The set system  $(E, \mathcal{F})$  is a closure space if it satisfies the following properties*

- (1)  $\emptyset \in \mathcal{F}, E \in \mathcal{F}$*
- (2)  $X, Y \in \mathcal{F}$  implies  $X \cap Y \in \mathcal{F}$ .*

Let a set system  $(E, \mathcal{F})$  be a closure space, then the operator

$$\tau(A) = \bigcap \{X : A \subseteq X \text{ and } X \in \mathcal{F}\} \quad (3)$$

is a closure operator.

Convex geometries were introduced by Edelman and Jamison [3] as a combinatorial abstraction of "convexity".

**Definition 1.6** [2] *The closure space  $(E, \mathcal{F})$  is a convex geometry if the family  $\mathcal{F}$  satisfies the following property*

$$X \in \mathcal{F} - E \text{ implies } X \cup x \in \mathcal{F} \text{ for some } x \in E - X. \quad (4)$$

It is easy to see that property (4) is dual to accessibility. Then, we will call it *up-accessibility*. If in each non-empty accessible set system one can reach the empty set  $\emptyset$  from any feasible set  $X \in \mathcal{F}$  by moving down, so in each non-empty up-accessible set system  $(E, \mathcal{F})$  the set  $E$  may be reached by moving up.

It is clear that a complement set system  $(E, \overline{\mathcal{F}})$  (system of complements), where  $\overline{\mathcal{F}} = \{X \subseteq E : E - X \subseteq \mathcal{F}\}$ , is up-accessible if and only if the set system  $(E, \mathcal{F})$  is accessible.

In fact, accessibility means that for all  $X \in \mathcal{F}$  there exists a chain  $\emptyset = X_0 \subset X_1 \subset \dots \subset X_k = X$  such that  $X_i = X_{i-1} \cup x_i$ ,  $x_i \in X_i - X_{i-1}$  and  $X_i \in \mathcal{F}$  for  $0 \leq i \leq k$ , and up-accessibility implies the existence of the corresponding chain  $X = X_0 \subset X_1 \subset \dots \subset X_k = E$ . Consider a set family for which this *chain property* holds for each pair of sets  $X \subset Y$ .

**Definition 1.7** *A set system  $(E, \mathcal{F})$  satisfies the chain property if for all  $X, Y \in \mathcal{F}$ , and  $X \subset Y$ , there exists an  $y \in Y - X$  such that  $Y - y \in \mathcal{F}$ . We call the system a chain system.*

In other words, a set system  $(E, \mathcal{F})$  satisfies the chain property if for all  $X, Y \in \mathcal{F}$ , and  $X \subset Y$ , there exists an  $y \in Y - X$  such that  $X \cup y \in \mathcal{F}$ .

**Proposition 1.8**  *$(E, \overline{\mathcal{F}})$  is a chain system if and only if  $(E, \mathcal{F})$  is a chain system as well.*

**Proof.** Let  $X, Y \in \overline{\mathcal{F}}$ , and  $X \subset Y$ , then there exist  $\overline{X} = E - X$  and  $\overline{Y}$  such that  $\overline{Y} \subset \overline{X}$ , and there is  $y \in \overline{X} - \overline{Y}$  such that  $\overline{Y} \cup y \in \mathcal{F}$ . Since  $\overline{X} - \overline{Y} = \overline{X} \cap Y = Y - X$ , we have  $y \in Y - X$ . In addition,  $\overline{Y} \cup y \in \mathcal{F}$  implies  $Y - y \in \overline{\mathcal{F}}$ , that completes the proof. ■

Consider a relation between accessibility and the chain property. If  $\emptyset \in \mathcal{F}$ , then accessibility follows from the chain property. In general case, there are accessible set systems that do not satisfy the chain property (for example, consider  $E = \{1, 2, 3\}$  and  $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, \{1, 2, 3\}\}$ ) and vice versa, it is possible to construct a set system, that satisfies the chain property and it is not an accessible (for example, let now  $\mathcal{F} = \{\{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ ). In fact, if we have an accessible set system satisfying the chain property, then the same system but without the empty set (or without all subsets of cardinality less than some  $k$ ) is not accessible, but satisfies the chain property. The analogy statements are correct for up-accessibility.

Examples of chain systems include convex geometries (see proposition 1.11) and their complement systems called antimatroids, hereditary systems (matroids, matchings, cliques, independent sets of a graph).

Consider another example of a chain system.

**Example 1.9** For a graph  $G = (V, E)$ , the set system  $(V, \mathcal{S})$  given by

$$\mathcal{S} = \{A \subseteq V : (A, E(A)) \text{ is a connected subgraph of } G\},$$

is a chain system. The example is illustrated in Figure 1.

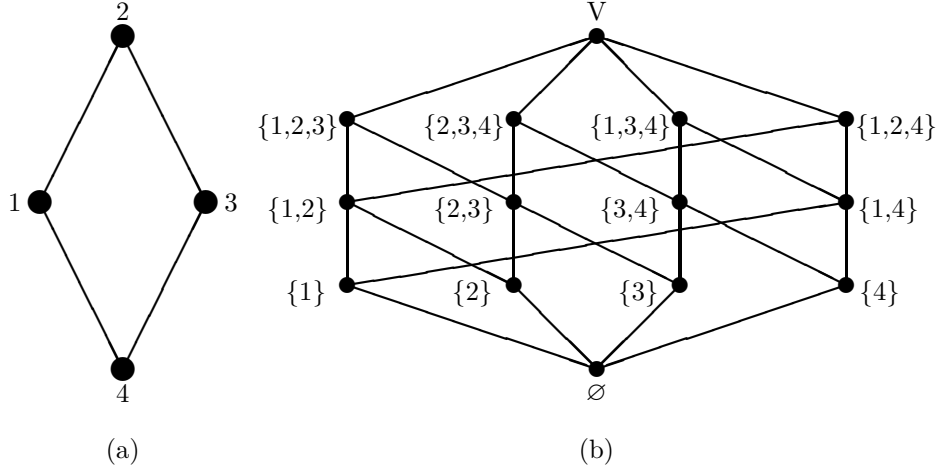


Figure 1:  $G = (V, E)$  (a) and a family of connected subgraphs (b).

To show that  $(V, \mathcal{S})$  is a chain system consider some  $A, B \in \mathcal{S}$  such that  $A \subset B$ . We are to prove that there exists an  $b \in B - A$  such that  $A \cup b \in \mathcal{S}$ . Since  $B$  is a connected subgraph, there is an edge  $e = (a, b)$ , where  $a \in A$  and  $b \in B - A$ . Hence,  $A \cup b \in \mathcal{S}$ .

For a set  $X \in \mathcal{F}$ , let  $ex(X) = \{x \in X : X - x \in \mathcal{F}\}$  be the set of *extreme points* of  $X$ . Originally, this operator was defined for closure spaces [3]. Our definition does not demand the existing of a closure operator, but when the set system  $(E, \mathcal{F})$  is a convex geometry  $ex(X)$  becomes the classical set of extreme points of a convex set  $X$ .

Note, that accessibility means that for each non-empty  $X \in \mathcal{F}$ ,  $ex(X) \neq \emptyset$ .

**Definition 1.10** The operator  $ex : \mathcal{F} \rightarrow 2^E$  satisfies the *heritage property* if  $X \subseteq Y$  implies  $ex(Y) \cap X \subseteq ex(X)$  for all  $X, Y \in \mathcal{F}$ .

We choose the name *heritage property* following B.Monjardet [12]. This condition is well-known in the theory of choice functions where one uses also alternative terms like *Chernoff condition* [1] or *property  $\alpha$*  [15]. This property is also known in the form  $X - ex(X) \subseteq Y - ex(Y)$ .

The heritage property means that  $Y - x \in \mathcal{F}$  implies  $X - x \in \mathcal{F}$  for all  $X, Y \in \mathcal{F}$  with  $X \subseteq Y$  and for all  $x \in X$ .

The extreme point operator of a closure space satisfies the heritage property, but the opposite statement is not correct. Indeed, consider the following example illustrated in Figure 2 (a): let  $E = \{1, 2, 3, 4\}$  and

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, E\}.$$

It is easy to check that the extreme point operator  $ex$  satisfies the heritage property, but the set system  $(E, \mathcal{F})$  is not a closure space ( $\{2, 4\} \cap \{3, 4\} \notin \mathcal{F}$ ). It may be mentioned that this set system does not satisfy the chain property. Another example (Figure 2 (b)) shows that the chain property is also not enough for a set system to be a closure space. Here

$$\mathcal{F} = \{\emptyset, \{1\}, \{4\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, E\},$$

and the constructed set system satisfies the chain property, but is not a closure set ( $\{1, 3\} \cap \{3, 4\} \notin \mathcal{F}$ ).

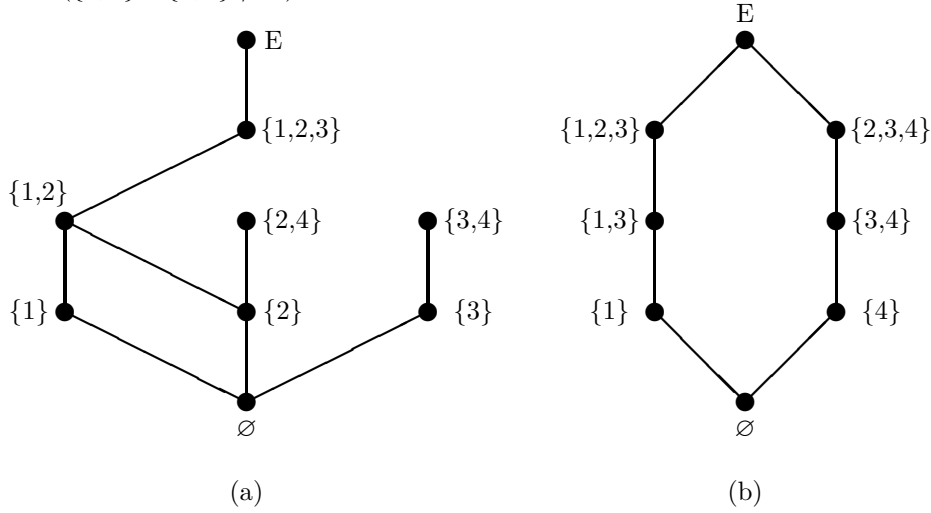


Figure 2: Heritage property (a) and chain property (b).

**Proposition 1.11** *A set system  $(E, \mathcal{F})$  is a convex geometry if and only if*

- (1)  $\emptyset \in \mathcal{F}$ ,  $E \in \mathcal{F}$
- (2) *the set system  $(E, \mathcal{F})$  satisfies the chain property*
- (3) *the extreme point operator  $ex$  satisfies the heritage property.*

**Proof.** Let a set system  $(E, \mathcal{F})$  be a convex geometry. Then the first condition automatically follows from the convex geometry definition. Prove the second condition. Consider  $X, Y \in \mathcal{F}$ , and  $X \subset Y$ . From (4) follows that there is a chain

$$X = X_0 \subset X_1 \subset \dots \subset X_k = E$$

such that  $X_i = X_{i-1} \cup x_i$  and  $X_i \in \mathcal{F}$  for  $0 \leq i \leq k$ . Let  $j$  be the least integer for which  $X_j \supseteq Y$ . Then  $X_{j-1} \not\supseteq Y$ , and  $x_j \in Y$ . Thus,  $Y - x_j = Y \cap X_{j-1} \in \mathcal{F}$ . Since  $x_j \notin X$ , the chain property is proved. To prove that  $ex(Y) \cap X \subseteq ex(X)$ , consider  $p \in ex(Y) \cap X$ , then  $Y - p \in \mathcal{F}$  and  $X \cap (Y - p) = X - p \in \mathcal{F}$ , i.e.,  $p \in ex(X)$ .

Conversely, let us prove that the set system  $(E, \mathcal{F})$  is a convex geometry. We are to prove both up-accessibility and that  $X, Y \in \mathcal{F}$  implies  $X \cap Y \in \mathcal{F}$ . Since  $E \in \mathcal{F}$ , up-accessibility follows from the chain property.

Consider  $X, Y \in \mathcal{F}$ . Since  $E \in \mathcal{F}$ , the chain property implies that there is a chain

$$X = X_0 \subset X_1 \subset \dots \subset X_k = E$$

such that  $X_i = X_{i-1} \cup x_i$  and  $X_i \in \mathcal{F}$  for  $0 \leq i \leq k$ . If  $j$  is the least integer for which  $X_j \supseteq Y$ , then  $X_{j-1} \not\supseteq Y$ , and  $x_j \in Y$ . Since  $x_j \in \text{ex}(X_j)$ , we obtain  $x_j \in \text{ex}(Y)$ . Continuing the process of clearing  $Y$  from the elements that are absent in  $X$ , eventually we reach the set  $X \cap Y \in \mathcal{F}$ . ■

## 2 Main results

In this section we consider relationship between quasi-concave set functions and monotone linkage functions.

Monotone linkage functions were introduced by Joseph Mullat [14].

A function  $\pi : E \times 2^E \rightarrow \mathbf{R}$  is called a *monotone linkage function* if

$$X \subseteq Y \text{ implies } \pi(x, X) \leq \pi(x, Y), \text{ for each } X, Y \subseteq E \text{ and } x \in E. \quad (5)$$

Consider function  $F : (2^E - \{E\}) \rightarrow \mathbf{R}$  defined as follows

$$F(X) = \min_{x \in X} \pi(x, X). \quad (6)$$

**Example 2.1** Consider a graph  $G = (V, E)$ , where  $V$  is a set of vertices and  $E$  is a set of edges. Let  $\text{deg}_H(x)$  denote the degree of vertex  $x$  in the induced subgraph  $H \subseteq G$ . It is easy to see that function  $\pi(x, H) = \text{deg}_H(x)$  is monotone linkage function and function  $F(H)$  returns the minimal degree of subgraph  $H$ .

**Example 2.2** Consider a proximity graph  $G = (V, E, W)$ , where  $w_{ij}$  represents the degree of similarity of objects  $i$  and  $j$ . A higher value of  $w_{ij}$  reflects a higher similarity of objects  $i$  and  $j$ . Define a monotone linkage function  $\pi(i, H) = \sum_{j \in H} w_{ij}$ , that measures proximity between subset  $H \subseteq V$  and their element  $i$ . Then the function  $F(H) = \min_{i \in H} \pi(i, H)$  can be interpreted as a measure of density of set  $H$ .

It was shown [9], that for every monotone linkage function  $\pi$ , function  $F$  is quasi-concave on the Boolean  $2^E$ . Moreover, each quasi-concave function may be defined by a monotone linkage function. In this section we investigate this relation on different families of sets.

For each function  $F$  defined on a set system  $(E, \mathcal{F})$ , we can construct the corresponding linkage function

$$\pi_F(x, X) = \begin{cases} \max_{A \in [x, X]_{\mathcal{F}}} F(A), & x \in X \text{ and } [x, X]_{\mathcal{F}} \neq \emptyset \\ \min_{A \in \mathcal{F}} F(A), & \text{otherwise} \end{cases}. \quad (7)$$

where  $[x, X]_{\mathcal{F}} = \{A \in \mathcal{F} : x \in A \text{ and } A \subseteq X\}$ .

**Proposition 2.3**  $\pi_F$  is monotone.

**Proof.** Indeed, if  $x \in X$  and  $[x, X]_{\mathcal{F}} \neq \emptyset$ , then  $X \subseteq Y$  implies  $[x, Y]_{\mathcal{F}} \neq \emptyset$  and

$$\pi_F(x, X) = \max_{A \in [x, X]_{\mathcal{F}}} F(A) \leq \max_{A \in [x, Y]_{\mathcal{F}}} F(A) = \pi_F(x, Y).$$

If  $x \in X$  and  $[x, X]_{\mathcal{F}} = \emptyset$ , then  $X \subseteq Y$  implies  $\pi_F(x, X) = \min_{A \in \mathcal{F}} F(A) \leq \pi_F(x, Y)$ . It is easy to verify the remaining cases. ■

Let  $(E, \mathcal{F})$  be an accessible set system. Denote  $\mathcal{F}^+ = \mathcal{F} - \emptyset$ . Then, having the linkage function  $\pi_F$ , we can construct for all  $X \in \mathcal{F}^+$  the set function

$$G_F(X) = \min_{x \in ex(X)} \pi_F(x, X). \quad (8)$$

Now consider the relationship between two set functions  $F$  and  $G_F$ .

**Proposition 2.4** If  $(E, \mathcal{F})$  is an accessible set system, then

$$G_F(X) \geq F(X), \text{ for each } X \in \mathcal{F}^+.$$

**Proof.** Indeed,

$$G_F(X) = \min_{x \in ex(X)} \pi_F(x, X) = \pi_F(x^*, X) = \max_{A \in [x^*, X]_{\mathcal{F}}} F(A) \geq F(X),$$

where  $x^* \in \arg \min_{x \in ex(X)} \pi_F(x, X)$ <sup>1</sup>. ■

What conditions on the set system  $(E, \mathcal{F})$  are to be satisfied to be sure that  $G_F$  coincides with  $F$ ?

**Theorem 2.5** Let  $(E, \mathcal{F})$  be an accessible set system. Then for every quasi-concave set function  $F : \mathcal{F}^+ \rightarrow \mathbf{R}$

$$G_F = F \text{ on } \mathcal{F}^+$$

if and only if the set system  $(E, \mathcal{F})$  satisfies the chain property.

**Proof.** Assume that the set system  $(E, \mathcal{F})$  satisfies the chain property. For each  $X \in \mathcal{F}^+$

$$G_F(X) = \min_{x \in ex(X)} \pi_F(x, X) = \min_{x \in ex(X)} F(A^x),$$

where  $A^x$  is a set from  $[x, X]_{\mathcal{F}}$  on which the value of the function  $F$  is maximal, i.e.,

$$A^x \in \arg \max_{A \in [x, X]_{\mathcal{F}}} F(A).$$

Consider  $Z$  that is a cover of  $\bigcup_{x \in ex(X)} A^x$ , i.e.  $Z \in \mathcal{C}(\bigcup_{x \in ex(X)} A^x)$ . From quasi-concavity (2) it follows that  $\min_{x \in ex(X)} F(A^x) \leq F(Z)$ . So,  $G_F(X) \leq F(Z)$  for

---

<sup>1</sup>arg min  $f(x)$  denote the set of arguments that minimize the function  $f$ .

each  $Z \in \mathcal{C}(\bigcup_{x \in ex(X)} A^x)$ . Now, to prove that  $G_F = F$ , it is enough to show that  $X \in \mathcal{C}(\bigcup_{x \in ex(X)} A^x)$ .

In fact, the stronger proposition is correct. If  $(E, \mathcal{F})$  is an accessible chain system, then for all  $X \in \mathcal{F}$  and  $B^x \in [x, X]_{\mathcal{F}}$

$$X \in \mathcal{C}(\bigcup_{x \in ex(X)} B^x). \quad (9)$$

For each  $x \in ex(X)$ ,  $X \supseteq B^x$ , and then  $X \supseteq \bigcup_{x \in ex(X)} B^x$ . Assume, that  $X$  is not a cover of  $\bigcup_{x \in ex(X)} B^x$ , i.e., there is a set  $Y$ , such that  $Y \in \mathcal{C}(\bigcup_{x \in ex(X)} B^x)$  and  $X \supset Y$ . Then from the chain property it follows that there exists an element  $y \in X - Y$  such that  $X - y \in \mathcal{F}$ , i.e., there exists  $y \in ex(X)$  and  $y \notin Y$ . On the other hand,

$$Y \in \mathcal{C}(\bigcup_{x \in ex(X)} B^x) \Rightarrow Y \supseteq \bigcup_{x \in ex(X)} B^x \supseteq ex(X),$$

contradiction that proves (9). Therefore,  $G_F(X) \leq F(X)$ , and, with (2.4),  $F = G$ .

Conversely, assume that the set system  $(E, \mathcal{F})$  does not satisfy the chain property. Since the set system  $(E, \mathcal{F})$  is an accessible system, it means that there exist  $A, B \in \mathcal{F}$  such that  $A \subset B$ ,  $A \neq \emptyset$  and there is not any  $b \in B - A$  such that  $B - b \in \mathcal{F}$ , i.e.,  $ex(B) \subseteq A$ .

It is easy to see that the function

$$F(X) = \begin{cases} 1, & X = A \\ 0, & \text{otherwise} \end{cases}.$$

is quasi-concave.

Consider the linkage function  $\pi_F$ . Since  $x \in ex(B)$  implies  $x \in A$ , then

$$\pi_F(x, B) = \max_{X \in [x, B]_{\mathcal{F}}} F(X) = F(A) = 1$$

Thus,  $G_F(B) = 1$ , i.e.  $G_F \neq F$ . ■

Thus, we proved that on an accessible set system satisfying the chain property each quasi-concave function  $F$  determines a monotone linkage function  $\pi_F$ , and a set function defined as a minimum of this monotone linkage function  $\pi_F$  coincides with the original function  $F$ .

As examples of such set system may be considered greedoids [2] that include matroids and antimatroids, and antigreedoids including convex geometries. By an antigreedoid we mean a set system  $(E, \mathcal{F})$  such that the complementary set system  $(E, \overline{\mathcal{F}})$  is a greedoid.

Note, that if  $F$  is not quasi-concave, the function  $G_F$  does not necessarily equal  $F$ . For example, let  $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$  and let

$$F(X) = \begin{cases} 0, & X = \{1, 2\} \\ 1, & \text{otherwise} \end{cases}$$



Function  $F$  is not quasi-concave, since  $F(\{1\} \cup \{2\}) < \min(F\{1\}, F\{2\})$ . It is easy to check that here  $G_F \neq F$ , because  $\pi_F(1, \{1, 2\}) = \pi_F(2, \{1, 2\}) = 1$ , and so  $G_F(\{1, 2\}) = 1$ . Moreover, the function  $G_F$  is quasi-concave. To understand this phenomenon, consider the opposite process.

Let  $(E, \mathcal{F})$  be an accessible set system. We can construct the set function  $F_\pi : \mathcal{F}^+ \rightarrow \mathbf{R}$  :

$$F_\pi(X) = \min_{x \in ex(X)} \pi(x, X), \quad (10)$$

based on the monotone linkage function  $\pi$  defined on  $E \times 2^E$ . To extend this function to the whole set system  $(E, \mathcal{F})$  define

$$F_\pi(\emptyset) = \min_{(x, X)} \pi(x, X).$$

**Theorem 2.6** *Let  $(E, \mathcal{F})$  be an accessible set system. Then the following statements are equivalent*

- (i) *the extreme point operator  $ex : \mathcal{F} \rightarrow 2^E$  satisfies the heritage property.*
- (ii) *for every monotone linkage function  $\pi$  the function  $F_\pi$  is quasi-concave.*

**Proof.** Let the extreme point operator  $ex$  satisfies the heritage property. To prove that the function  $F_\pi$  is a quasi-concave function on  $\mathcal{F}$ , first note that

$$Z \in \mathcal{C}(X) \text{ implies } ex(Z) \subseteq X \text{ for each nonempty } X \subseteq E. \quad (11)$$

This statement immediately follows from the definition of a cover set.

Consider some  $Z = \mathcal{C}(X \cup Y)$ . Let  $F_\pi(Z) = \min_{x \in ex(Z)} \pi(x, Z)$ . Then  $F_\pi(Z) = \pi(x^*, Z)$ , where  $x^* \in \arg \min_{x \in ex(Z)} \pi(x, Z)$ . Then, by (11),  $x^* \in X \cup Y$ . Assume, without loss of generality, that  $x^* \in X$ . Thus by the heritage property  $x^* \in ex(X)$ , because  $x^* \in X$ , and  $X \subseteq Z$ , and  $x^* \in ex(Z)$ . Hence

$$F_\pi(Z) = \pi(x^*, Z) \geq \pi(x^*, X) \geq \min_{x \in ex(X)} \pi(x, X) = F_\pi(X) \geq \min\{F_\pi(X), F_\pi(Y)\}.$$

Conversely, assume that the extreme point operator  $ex$  does not satisfy the heritage property, i.e., there exist  $A, B \in \mathcal{F}$  such that  $A \subset B$ , and there is  $a \in A$  such that  $B - a \in \mathcal{F}$  and  $A - a \notin \mathcal{F}$ .

It is easy to check that the function

$$\pi(x, X) = \begin{cases} 1, & x = a \\ 2, & \text{otherwise} \end{cases} .$$

is monotone.

Then,  $F_\pi(B) = 1$ ,  $F_\pi(A) = F_\pi(B - a) = 2$ . Since  $A \cup (B - a) = B$ , we have

$$F_\pi(A \cup (B - a)) < \min\{F_\pi(A), F_\pi(B - a)\}$$

i.e.,  $F_\pi$  is not a quasi-concave function. ■

Thus, if a set system  $(E, \mathcal{F})$  is accessible and the operator  $ex$  satisfies the heritage property, then for each set function  $F$ , defined on  $(E, \mathcal{F})$ , one can build the quasi-concave set function  $G_F$  that is an upper bound of the original function  $F$ .

We show the corresponding property holds also for monotone linkage functions.

**Theorem 2.7** *Let  $(E, \mathcal{F})$  be an accessible set system with the operator  $ex$  satisfying the heritage property, and let a function  $F_\pi$  be defined as a minimum of a monotone linkage function  $\pi$  by (10), then  $\pi_F|_{\mathcal{F}} \leq \pi|_{\mathcal{F}}$ , i.e., for all  $X \in \mathcal{F}$  and  $x \in ex(X)$*

$$\pi_F(x, X) \leq \pi(x, X),$$

where  $\pi_F$  is defined by (7).

**Proof.** For all  $X \in \mathcal{F}$  and  $x \in ex(X)$

$$\pi_F(x, X) = \max_{A \in [x, X]_{\mathcal{F}}} F(A) = F(A^x) = \min_{a \in ex(A^*)} \pi(a, A^x) \leq \pi(x, A^x),$$

where  $A^x \in \arg \max_{A \in [x, X]_{\mathcal{F}}} F(A)$ .

The last inequality follows from the heritage property. Indeed,  $X \supseteq A^x$  and  $x \in A^x$ , then  $x \in ex(X)$  implies  $x \in ex(A^x)$ .

Now, from monotonicity of the function  $\pi$  we have  $\pi(x, A^x) \leq \pi(x, X)$ , that finishes the proof. ■

Consider the following example to see that the functions  $\pi$  and  $\pi_F$  can be not equal. Let  $E = \{1, 2\}$ ,  $\mathcal{F} = 2^E$ .

$$\pi(x, X) = \begin{cases} 2, & x = 2 \text{ and } X = \{1, 2\} \\ 1, & \text{otherwise} \end{cases}$$

then the function  $F(X) = \min_{x \in ex(X)} \pi(x, X)$  is equal to 1 for all  $X \subset E$ , and then

$\pi_F$  is equal for 1 for each pair  $(x, X) \in E \times 2^E$ , i.e.,  $\pi_F \neq \pi$ .

Define more exactly the structure of the set of monotone linkage functions.

**Theorem 2.8** *Let  $(E, \mathcal{F})$  be an accessible set system, and let  $\pi_1$  and  $\pi_2$  define (by (10)) the same set function  $F$  on  $\mathcal{F}$ . Then the function*

$$\pi = \min\{\pi_1, \pi_2\}$$

*is a monotone linkage function determines the same function  $F$  on  $\mathcal{F}$ .*

**Proof.** At first, prove that  $\pi$  is a monotone linkage function. Indeed, consider a pair  $X \subseteq Y$ . Without loss of generality we have

$$\pi(x, Y) = \min\{\pi_1(x, Y), \pi_2(x, Y)\} = \pi_1(x, Y).$$

Then, from monotonicity,

$$\pi_1(x, Y) \geq \pi_1(x, X) \geq \min\{\pi_1(x, X), \pi_2(x, X)\} = \pi(x, X)$$

Now, denote  $G(X) = \min_{x \in ex(X)} \pi(x, X)$  and prove that  $G = F$ .

We have

$$G(X) = \min_{x \in ex(X)} \pi(x, X) = \pi(x^*, X) = \min\{\pi_1(x^*, X), \pi_2(x^*, X)\},$$

where  $x^* \in \arg \min_{x \in ex(X)} \pi(x, X)$ . Without loss of generality we have

$$G(X) = \pi_1(x^*, X) \geq \min_{x \in ex(X)} \pi_1(x, X) = F(X).$$

On the other hand,

$$F(X) = \min_{x \in ex(X)} \pi_1(x, X) = \pi_1(x^\#, X) \geq \pi(x^\#, X) \geq \min_{x \in ex(X)} \pi(x, X) = G(X).$$

■

Thus, the set of monotone linkage functions, defined by the set function  $F$  on an accessible set system, forms a semilattice with the lattice operation

$$\pi_1 \wedge \pi_2 = \min\{\pi_1, \pi_2\},$$

where the function  $\pi_F$  is a null of this semilattice (follows from Theorem 2.7).

### 3 Conclusion

Some aspects of duality between quasi-concave set functions and monotone linkage functions were discussed for convex geometries, and more generally, for chain systems.

Our findings may lead to efficient optimization procedures on more complex set systems than just matroids and antimatroids.

### References

- [1] H. Chernoff, *Rational selection of decision functions*, *Economica* 22, (1954) 422-443.
- [2] B. Korte, L. Lovász, and R. Schrader, *"Greedoids"*, Springer-Verlag, New York/Berlin, 1991.
- [3] P.H. Edelman and R.E. Jamison, *The theory of convex geometries*, *Geom. Dedicata*, 19, (1985) 247-270.
- [4] J. Edmonds, *Matroid and the greedy algorithm*, *Mathematical Programming* 1, (1971) 127-136.
- [5] Y. Kempner, and V.E. Levit, *Correspondence between two antimatroid algorithmic characterizations*, *The Electronic Journal of Combinatorics* 10 (2003) R44.

- [6] Y. Kempner, B. Mirkin, and I. Muchnik, *Monotone linkage clustering and quasi-concave functions*, [Applied Mathematics Letters](#) **10** , No.4 (1997) 19-24.
- [7] Y. Kempner and I. Muchnik, *Clustering on antimatroids and convex geometries*, WSEAS Transactions on Mathematics, 2, Issue 1 (2003) 54-59.
- [8] Y. Kempner and I. Muchnik, *Quasi-concave functions on meet-semilattices*, Discrete Applied Mathematics **156**, No. 4, (2008) 492-499.
- [9] A. Malishevski, *Properties of ordinal set functions*, in A.Malishevski, In the book "Qualitative Models in the Theory of Complex Systems", Nauka, Moscow, (1998) 169-173 (in Russian).
- [10] B.Mirkin and I.Muchnik, *Layered clusters of tightness set functions*, [Applied Mathematics Letters](#), **15** (2002) 147-151.
- [11] B.Mirkin and I.Muchnik, *Induced layered clusters, hereditary mappings, and convex geometry*, [Applied Mathematics Letters](#), **15**, (2002) 293-298.
- [12] B. Monjardet and V. Raderanirina, *The duality between the anti-exchange closure operators and the path independent choice operators on a finite set*, Mathematical Social Sciences **41**, (2001) 131-150.
- [13] I. Muchnik and L.V. Shvartser, *Kernels of monotonic systems on a semi-lattice of sets*, Automation and Remote Control, Vol. 50, No. 8, part 2, (1989) 1095-1102.
- [14] J. Mullat, *Extremal subsystems of monotone systems: I, II*, Automation and Remote Control **37**, (1976) 758-766; 1286-1294.
- [15] A.K. Sen, *Choice functions and revealed preference*, Review of Economic Studies **38**, (1971) 307-317.
- [16] Y. Zaks (Kempner) and I. Muchnik, *Incomplete classifications of a finite set of objects using monotone systems*, Automation and Remote Control **50**, (1989) 553-560.