QUASI-CONCAVE FUNCTIONS ON POLY-ANTIMATROIDS

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Abstract: Our work is devoted to investigation of "multi" versions of antimatroids (dual of convex geometries). These discrete structures are present in many fields of mathematical social sciences. For instance, we can quote the theory of choice, where a first attempt to connect choice functions and closure operators appears in (Malishevski, 1994). Recently, this link was interpreted as duality between path-independent choice functions and convex geometries (Johnson, Dean, 2001), (Koshevoy, 1999), (Monjardet, Raderanirina, 2001). Another example is game theory where antimatroids are considered as permission structures for coalitions (Bilbao, 2003).

Multisets also arise in various areas of mathematics and computer science. The combination of notions of an antimatroid and a multiset, namely, an antimatroid with repetitions, was proposed by Bjorner, Lovasz and Shor in 1991 in order to analyze the chip game.

In this paper we define a poly-antimatroid as a family of multisets, and investigate the structure of quasi-concave functions on poly-antimatroids.

1. Introduction.

Let E be a finite set.

A *multiset* A over E is a function $f_A : E \to N$, where $f_A(e)$ is the number of repetition of an element e in A.

A *multiset system* over E is a pair $(E, \Im)$, where $\Im$ is a family of multisets over E, called *feasible* multisets.

The following definition of a poly-antimatroid is identical with the definition of an antimatroid except that the set system is replaced by the system of multisets.

A finite non-empty multiset system $(E, \Im)$ is a *poly-antimatroid* if

(*A1*) for each non-empty $X \in \Im$ there is an $x \in \Im$ such that $X - x \in \Im$

(*A2*) for all $X, Y \in \Im$, and $X \not\subset Y$, there exist an $x \in X - Y$ such that $Y \cup x \in \Im$.

Now consider the definition of a poly-antimatroid as a formal language given in (Bjorner, Lovasz, Shor, 1991). A *word* over E is a finite sequence formed from the elements (letters) of E. A *language* L over E is a set of words over E. The concatenation of two words α and β will be denoted $\alpha\beta$, and the multiset of letters in a word α will be denoted by $\tilde{\alpha}$.

An *antimatroid with repetitions* is a language (E,L) satisfying the following three properties:
(i) left-hereditary: If $\alpha x \in L$, then $\alpha \in L$.
(ii) locally free: Let $\alpha \in L$ and $x \neq y$ be two letters from E such that $\alpha x \in L$ and $\alpha y \in L$, then $\alpha xy \in L$.
(iii) permutable: If $\alpha, \beta \in L$, $\tilde{\alpha} = \tilde{\beta}$, and $\alpha x \in L$ for some $x \in E$, then $\beta x \in L$.

We prove that poly-antimatroids and antimatroids with repetitions are equivalent in the following sense.

Theorem 1.1. *If* (E,L) *is an antimatroid with repetitions, then* $P(L) = \{\tilde{\alpha} : \alpha \in L\}$ *is a poly-antimatroid* $(E, P(L))$.
Conversely, if $(E,\Im)$ *is a poly-antimatroid, then* $l(\Im) = \{x_1 ... x_k : \{x_1 ... x_j\} \in \Im \text{ for } 1 \le j \le k\}$ *is an antimatroid with repetitions* $(E, l(\Im))$. *Moreover,* $l(P(L)) = L$ *and* $P(l(\Im)) = \Im$.

Further we consider the quasi-concave functions on poly-antimatroids.

A function F defined on a poly-antimatroid $(E,\Im)$ is *quasi-concave* if for each $X, Y \in \Im$, $F(X \wedge Y) \ge \min\{F(X), F(Y)\}$, where $X \wedge Y$ is the maximal feasible sub-multiset of $X \cap Y$.

Originally, these functions were considered on the boolean 2^E, where the last inequality turns into the following one $F(X \cap Y) \ge \min\{F(X), F(Y)\}$. In (Malishevski, 1998) a characterization of quasi-concave functions defined on 2^E was established. In our work we extend these results to poly-antimatroids.

2. Main results.

We generalize the definition of a monotone linkage function, given for set systems in (Mullat, 1976), to multisets.

A function $\pi : E \times N^E \to R$ is called a *monotone linkage function* if for all $X, Y \in N^E$ and $x \in E$,

$$f_X(x) = f_Y(x) \text{ and } X \subseteq Y \text{ implies } \pi(x, X) \ge \pi(x, Y).$$

The following theorem characterizes quasi-concave functions defined on poly-antimatroids. Note, that in fact, we consider the functions defined on $\Im - E_\Im$, where $E_\Im$ is the maximal feasible multiset of $\Im$.

Theorem 2.1. *A function* F *defined on poly-antimatroid* $(E,\Im)$ *is quasi-concave if and only if there exists a monotone linkage function* π *such that for each* $X \in \Im - E_\Im$

$$F(X) = \min_{x \in \Gamma(X)} \pi(x, X),$$

where $\Gamma(X) = \{x \in E : X \cup x \in \Im\}$ *is a set of feasible continuations of* X.

The proof of the theorem is based on the fact that for each poly-antimatroid every quasi-concave function F determines a monotone linkage function:

$$\pi_F(x,X) = \begin{cases} \max\limits_{A\in S^x(X)} F(A), & S^x(X) \textit{ is not empty} \\ \min\limits_{A\in \Im - E_\Im} F(A), & \textit{otherwise} \end{cases},$$

where

$$S^x(X) = \{Y \in \Im - E_\Im : X \subseteq Y \textit{ and } f_X(x) = f_Y(x)\},$$

such that $F(X) = \min\limits_{x\in\Gamma(X)} \pi_F(x,X)$.

A weaker property holds for monotone linkage functions.

Theorem 2.2. *Let* $F(X) = \min\limits_{x\in\Gamma(X)} \pi(x,X)$ *for a monotone linkage function* π *on a poly-antimatroid* $(E,\Im)$, *then* $\pi_F(x,X) \le \pi(x,X)$ *for any* $X \in \Im - E_\Im$ *and* $x \in \Gamma(X)$.

Now let us define more exactly the structure of the set of monotone linkage functions.

Theorem 2.3. *The set of monotone linkage functions, defining a function* F *on a poly-antimatroid, forms a semilattice with the following lattice operations* $\pi_1 \wedge \pi_2 = \min\{\pi_1,\pi_2\}$, *where the function* π_F *is a null of this semilattice.*

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