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Application of Monotone Systems to the Study of the Structure of Markov Chains

Abstract. The method of analysis of "Markov chain" is described. The method is based on the transformation of the Markov chain into a monotonic system and on the separation of kernels from the transformed chain.
Keywords: Markov chain; communication line; network; transition matrix; kernel

1. Introduction

In the work presented here, the theory of monotonic systems developed in an earlier publication (Mullat, a) 1976) is applied to the Markov chains. In the study of Markov chains the interest stems from the fact that it is convenient to interpret a special class of absorbing chains as monotonic systems. On the other hand, it also provides a meaningful way of illustrating the main properties of monotonic systems, as shown here using an example based on communication networks.[1] In order to disclose on conceptual level the technology developed for extracting the extreme subsystems in Markov chains discussed in the current paper, we employ the communication network as an example of monotonic system, albeit in a slightly modified form relative to that originally proposed in the context of telephone network. This will enable us to elucidate the manner in which a Markov chain may be associated with the monotonic system and what principal operations may be performed on it towards utilization of monotonic systems theoretical apparatus described in the Mullat original work.

In the earlier paper on which this Mullat work is based, an example of a communication network has been considered, whereby a set W comprising of communication lines/channels between some nodes — communicating units — was introduced.[2] Here, we will assume that each line has certain built-in redundancy mechanisms, such as the main and the reserved channels.[3] Thus, if a direct line is not available between nodes, analogously to what was described in Mullat's work 1976, the traffic might be organized through pass-around channels. In addition to this mechanism, in the present case, the possibility of employing pass-around communication is not excluded even if a direct channel is available.

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[1] Translated from Russian, Mullat, c) 1979. In the original paper, the term used was "telephone switch net," which was not adopted here, as it is outdated. Still, the concept underpinning the work remains highly relevant, as forms of "switches" are still used in redirecting TCP/IP packages, in a manner comparable to the telephone net.

[2] Switch is a device of such type and can learn where to address the communication packages.

[3] In practice, network redundancy may be guaranteed by some additional channels/lines activated only in urgent situations when the net usage exceeds some predefined threshold.

In the example presented in the original paper (Mullat, 1976), an average number of "denials" before establishing the contact characterizing each pair of nodes was utilized. The number of denials usually characterizes the communication lines in the communication network.[4] In the model described below, and for the purpose of current investigation, it is more convenient to use a value inverse to the number of denials, as this will characterize the communication line throughput.

Let us assume that each communication line (comprising of both the main and the reserved channels) is characterized by the throughput $c_{i,j}$ or, in other words, by the maximum allowed bandwidth usage, expressed in kilobytes for example. The value $c_{i\,j}$ thus denotes the throughput of main and reserved channels. We then explicate the communication center s by the maximum permissible usage

$$c_i = \sum_{j=1}^{n} c_{i,j} \,.$$

The traffic redirected through the node s along the main communication channel, as well as the reserved channel, between nodes s and j specifies thereupon a share of maximum permitted usage c_s. In an actual communication network, the usage share must be lower than the maximum allowed share $p_{s\,j} = {c_{s,j}}/{c_s}$. Moreover, the usage share $p_{s,j}$ of the communication channel can be interpreted as a probability of establishing contact between the nodes s and j. Assuming that the main and the reserved channels are treated as equitable, the quantity must satisfy an inequality

$$2 \cdot \sum_{j=1}^{n} p_{i,j} < 1 \tag{1}$$

for all s without exception,

Let a communication network, characterized by the aforementioned pass-around traffic feasibility, function during a long period of time by originating its main channels. We can characterize the traffic along each main channel (more precisely, the nodes i and j) by the average number of hits $\overline{p}_{i,j}$ that occur in the process of establishing either direct or indirect (pass-around) contact. It is apparent that $\overline{p}_{i,j}$ is slightly greater than the corresponding $p_{i,j}$.

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4 Network protocol analyzers can collect such types of statistical data.

If a malfunction occurs somewhere along the channel, the change[5] in the communication network will be reflected in a decrease in $\overline{p}_{i,j}$. In such a scenario, activating a reserved channel can accommodate higher network usage. It is obvious that, in this case, all $\overline{p}_{i,j}$ values will increase accordingly. Organized in this manner, the communication network represents a monotonic system.

However, a problem arises with respect to identifying the type of change malfunctioning/activating of a main/reserved channel that would influence the $\overline{p}_{i,j}$ values. In order to find an appropriate solution, it is necessary to explain the problem in Markov chains nomenclature.

Consider a set W of communication channels described by a square matrix $\left\|p_{i,j}\right\|_n^n$, when no channels exist, $p_{i,j} = 0$. According to the theory of Markov chains (Chung 1960). Such matrices may be associated with a set of returning states for some absorbing Markov chain. In the nomenclature pertaining to chains of this type, the value $\overline{p}_{i\,j}$ can be interpreted as an average number of hits from node i into node j along the Markov chain. Similarly, a malfunction in the main channel, resulting in the activation of the reserved channels, can be described through recalculating the average hit values $\overline{p}_{i,j}$. The above can be denoted as an action of type $\ominus$, whereas in the nomenclature of monotonic systems, an action of type $\oplus$ pertains to activating the reserved channel due to the malfunctioning in the main channel.

From the above discussion, it is evident that adopting this special class of absorbing Markov chains allows approaching the problem from the perspective of how to differentiate the Extremal Subsystem of Monotonic System — the kernels. Along with the KSR — Kernel Search Routine elaborated for this purpose in (Mullat, 1976), this approach can actually accomplish the kernel search task.

In Section II below, the problem of kernel extraction on Markov chains is described in more detail. In Section III, we show that the results of performing the $\oplus$ and $\ominus$ actions upon Markov chain entries in a transition matrix lead to Sherman-Morrison (Dinkelbach, 1969) expressions for recalculating the numbers of average hits (see Appendix).

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5 For example, the OSPF (Open Short Path First) protocol will automatically redirect the traffic.

2. The problem of Kernel Extraction on Markov Chains

Consider a stationary Markov chain with a finite number of states and discrete time. We denote the set of states by V. Stationary Markov chain can be characterized by the property that the pass probability from the state i to the state j at a certain point in time $t+1$ does not depend upon the state s $(s = 1,2,...,n)$ the considered chain arrived in i in the preceding moment t. We denote by $p(i, j, k)$ $(p(i, j, 1) = p_{i,j})$ the conditional probability of this pass from i to j within k units of time.

Below, we consider only a special class of Markov chains that, for arbitrary states i and j within some subset in V, is constrained by

$$\lim_{k\to\infty} p(i, j, k) = 0 .$$

According to the theory of Markov chains, this limit equals zero when the state j is returning, implying that there must be some reversible states in such Markov chains. Without diminishing the generality of this consideration, we will further examine chains with only one reversible state, which must simultaneously be an absorbing state.

The absorbing chains utilized below satisfy the following properties:

1. There exist only one absorbing state $\theta \in V$
2. All remaining states are returning, and the probability of a pass between the states in one step corresponds to an entry in the square matrix $\left\| p_{i,j} \right\|_n^n$.
3. The probability of a pass into an absorbing state θ from some returning state i in one step, in accordance with 1 and 2, is equal to

$$p_{i\,\theta} = 1 - \sum_{\theta=1}^{n} p_{i,\theta} \cdot$$

The monotonic system mandates a definition of some positive and negative (⊕, ⊖) actions upon system elements. For this purpose, we make use of the average number of hits $\overline{p}_{i,j}$ from the state i into the state j along the chain (Chung 1960). It is known that the value of $\overline{p}_{i\,j}$ is specified by the series

$$\overline{p}_{i,j} = \sum_{k=1}^{\infty} p(i, j, k) . \qquad (2)$$

The sufficient condition for series (2) to converge is established if the sum of entries in each row of the matrix $\|p_{i,j}\|_n^n$ is less than one. We consider that elements elsewhere in the chains fulfill the conditions 1-3.

Let W be the set of all nonzero entries in the matrix $\|p_{i,j}\|$. On the transition W set of the Markov chain described above, we define the following actions.

Definition. The action type ⊖ on the element of the system W (nonzero element of the matrix $\|p_{i,j}\|$) denotes a decrease in its value by some Δp of its probability to pass in one step.

By analogy, we define the action ⊕. In this case, the probability of a pass in one step, which corresponds to the entry value $p_{i,j}$, is increased by Δp. In case of some nonzero increment in the matrix $\|p_{i,j}\|$ element (based on straightforward probability considerations), all average numbers of hits $\overline{p}_{i\,j}$ must also increase accordingly. On the other hand, a Δp decrement would result in a decrease in the corresponding $\overline{p}_{i\,j}$ values. In sum, introduced actions upon system W elements fully meet the monotonic condition (Mullat, 1976), and system W transforms into a monotonic system.

At this juncture, it is important to emphasize that the Δp changes in values of probabilities in one step within W are not specified in the definition of ⊕ and ⊖ actions upon the entries in the matrix $\overline{p}_{i,j}$. Relatively rich possibilities exist for the change definition. For example, it can denote an increase (decrease) in each probability on a certain constant, or the same change, but this time depending upon the probability value itself, etc. When providing the definitions of ⊕ and ⊖ actions on an absorbing Markov chain, it is desirable to utilize authentic considerations. Below, using an example of communication network, we describe one of such considerations.

Let W be the set of all possible transitions in one step among all returning states of an absorbing chain. These transitions in the set W retain the correspondence with nonzero elements of the matrix $\|p_{i,j}\|$. Let T be a certain subset of the set W, relating to the nonzero elements noted above. Denote by $p(T,i,j,k)$ the probability that the chain passes from the state i into the state j within k time units, constrained by the condition that, during this period, all passes in one step upon the set T have been changed by either ⊕ or ⊖ actions. This condition corresponds to the assertion that the passes along the set $W \setminus T \equiv \overline{T}$ proceed in accordance with the "old" probabilities, while those along T are in governed by the "new" Probabilities. We do not exclude the case when no ⊕ or ⊖ actions have been implicated — the set $T = \varnothing$. In this case, we simply omit the T symbol notation in the corresponding probabilities.[6]

The average number of hits from i into j, subject to the constraint that some passes in the set T have been changed by actions, is specified by a series

$$\overline{p}(T, i, j) = \sum_{m=1}^{\infty} p(T, i, j, m). \tag{3}$$

Let us now focus on the collections of credentials specified by a monotonic system W. We define a collection $\Pi^{+}H$ on the subset $H \in W$ as a collection of real numbers $\{\overline{p}(\overline{H}, i, j) \,|(i,j) \in H\}$ in case that the positive ⊕ actions occur on $\overline{H} = W \setminus H$, while $\Pi^{-}H = \{\overline{p}(\overline{H}, i, j) \,|(i,j) \in H\}$ collection corresponds to the case of the negative ⊖ actions taking place.

In the original paper (Mullat, 1976), we have proved that, in a monotonic system, two kinds of subsystems always exist — the ⊕ and ⊖ kernels. The definitions introduced above, pertaining to the average number of hits $\overline{p}(\overline{H}, i, j)$, allow us to formulate the notion of ⊕ and ⊖ kernels in the Markov chain.

Definition. By the Extremal Subsystem of passes on absorbing Markov chain — the ⊕ and ⊖ kernels — we call a system $H^{\oplus} \subseteq W$, on which the functional

$$\max_{(i,j)\in H} \overline{p}(\overline{H}, i, j) \tag{4}$$

reaches its global minimum on 2^{W}, whereby ⊖ kernels will be a subsystem $H^{\ominus} \subseteq W$ where the functional

$$\min_{(i,j)\in H} \overline{p}(\overline{H}, i, j) \tag{5}$$

reaches its global maximum as well.

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6 We suppose that actions do not violate the convergence of probability series, see condition (1).

We will now turn the focus toward the notions of ⊕ and ⊖ kernels introduced above, using an example on communication network described earlier.

The probabilities of hits p_{ij} (without any passes, i.e., in a single step) between nodes i and j $\left(i, j = \overline{1, n}\right)$ allow us to construct for the communication network an absorbing chain satisfying the conditions 1-3 above. In fact, as we already noted, only one condition is mandatory to satisfy the inequality (1), which is a natural condition for any communication network. Conditions 2 and 3, on the other hand, can be guaranteed by the Markov chain design. In this case, numbers $p_{i,j}$ may be interpreted as probabilities of a pass in one step, whereby $\overline{p}_{i,j}$ denotes an average number of hits from i into j, whether directly, or via an indirect pass-around along other lines in the chain.

The search for the ⊕ and ⊖ kernels on an actual Markov chain, reconstructed from a communication network, mandates a precise definition of ⊕ and ⊖ actions. In the beginning of the discussion, we observed that ⊖ action might represent a malfunctioning in the main channel, whereas ⊕ action might pertain to the activation of a reserved channel. On the Markov chain, the malfunctioning is denoted as null, reducing the corresponding probability, while the activating of a reserved channel is reflected in the doubling of its initial probability value.[7] The condition (1) guarantees that, in any circumstance that would necessitate such ⊕ and ⊖ actions, the convergence of series (2) and (3) will not be violated.

We suggest a suitable interpretation of ⊕ and ⊖ kernels in Markov chain below, starting from the Markov chain characteristics, introduced here in terms of communication network.

In Extreme Subsystem $H^{\ominus}$, none of the communication lines/channels are subject to changes, whereas in all lines outside $H^{\ominus}$, they're reserved channels have been activated. The extreme value of the functional (4) shows that the average number of hits within channels belonging to $H^{\ominus}$, including the indirect pass-around hits (by definition, an indirect hit requires at least two steps to reach the destination), is relatively low. This assertion implies that the lines within the $H^{\ominus}$ kernel are "immune" with respect to package delivery malfunctions, i.e., most of the transported packages pass along direct lines. The set of lines in $H^{\oplus}$ kernel is characterized by a reverse property. Thus, the main

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7 We stress once again that ⊕ and ⊖ actions are subjective evaluations of an actual situation.

channels in $H^{\ominus}$ kernel are the most "appropriate" for organizing "high-quality" indirect communications, but are also a sensible choice for mitigating the malfunctions that may result in a "snowballing" or "bandwagon" effects. Conversely, along $H^{\oplus}$, the indirect communication is typically hampered for some reason.

3. MONOTONE SYSTEM CREDENTIAL FUNCTIONS ON MARKOV CHAINS

In Section II, we defined some $\oplus$ and $\ominus$ actions upon the transition matrix entries in one step corresponding to returning states. In this section, we will develop an apparatus that allows us to incorporate the changes induced by these two types of actions into the average numbers of hits from one returning state i into the other state j. We describe here and deduce some tangible credential functions intended for use alongside our formal monotonic system description, following the conventions presented in the previous work (Mullat, 1976). Let us first recollect the notion of credential function before providing an account of the main section contents.

Suppose that, in the system W, which in the case of Markov chain is characterized as a collection of entries in matrix $\left\| p_{i,j} \right\|_n^n$ corresponding to passes among returning states, a subset H has been extracted. As a result, the set H consists of one-step transitions. Owing to the successive actions of type $\ominus$, by accounting for all individual sequential steps in the process (see Section II) taken upon the elements in $\overline{H}$ (a complementary of H to W), it is possible to establish the average number of hits within the transition set H — the credential system $\Pi^{-}H$. By analogy, on the set $\overline{H}$, a succession of $\oplus$ actions establishes the credential system $\Pi^{+}H$. The average number of hits in the nomenclature given in Section II may be represented as $\overline{p}(\overline{H}, i, j)$ — i.e., the limit values for series (2) on nonzero elements for the transition matrix P corresponding to the entries/lines within the set H. Further, we will refer to the numbers $\overline{p}(\overline{H}, i, j)$ as the credential functions.

Let us now establish the general form of the credential functions on Markov chains as a matrix series. This can explain the mechanism of actions the defined in Section II, performed upon the elements of a monotonic system — the Markov chain.

The credential function on Markov chain may be found using the series (2), where the single element (i, j) in the series presents the probability of the chain pass from i into j, constrained by the condition that actions have been performed upon the set $\overline{H}$.

The general matrix form of such transition probabilities described in Section II is given below: θ

$$\left\| \begin{matrix} 1 & 0 & . & . & . & 0 \\ p_{1,\theta} & & & & & \\ . & & P & & & \\ p_{n,\theta} & & & & & \end{matrix} \right\| \text{, where} \tag{6}$$

θ – absorbing state of the chain;

$p_{i,\theta}$ – the probability of a pass from the i's returning state into the absorbing state θ;

P – the transition matrix of probabilities between the returning states within one step, where the matrix dimension is $n \times n$.

Using Chapman-Kolmogorov equations (Chung 1960), the element $p(T, i, j, m)$ in series (3) may be found as the m-s power of the matrix (6), whereby it occupies an entry in the matrix P^m.

In summary, the collection of series (3) may be written as the following matrix series

$$\overline{P}_T = I + P_T + P_T^2 + . . .,^8 \tag{7}$$

P_T – the matrix, where type ⊕ and ⊖ actions have been performed upon all nonzero elements within the set. Recall that, in the definition of a monotonic system, the credential function on the set $H \subseteq W$ takes advantage of a complementary set $\overline{H}$ to the set H only. The set $\overline{H}$ is actually the set of performed actions. Given that the elements of the set W are also presented as matrix entries $\overline{P}_{\overline{H}} = \left\| I - P_{\overline{H}} \right\|^{-1}$, the matrix is the credential functions collection on the Markov chain, identical to the matrix limit of (7).

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[8] We suppose that $p(T, i, j, 0) = \delta_{i,j}$, which is what the unity matrix in Section I highlights. In the nomenclature of the Markov chains (Kemeny et al, 1976) theory, matrices of type P_T are referred to as the fundamental matrices.

In the nomenclature of fundamental matrices, the actions upon the monotonic system elements are transformations, taking place in succession, from the matrix $\left\|I - P_T\right\|^{-1}$ to the matrix $\left\|I - P_{T\cup\alpha}\right\|^{-1}$. Calculus of such a transformation is, however, a very "hard operation." In order to organize the search of ⊕ and ⊖ kernels on the basis of constructive procedures (KSR) described previously (Mullat, 1976), the utilization of matrix form is inappropriate. To extract the extreme subsystems on Markov chains successfully and take full advantage of the developed theory of monotonic systems, a more effective technology is needed, which leads us to Sherman-Morrison relationships (Dinkelbach, 1969).

The solution that can account for the changes emerging as a result of the ⊕ and ⊖ actions upon the transition matrix elements within one step in the fundamental matrix of Markov chain may be archived in the following manner. Suppose that, instead of the old probability p_o denoting a pass in between the returning states i and j, an updated (new) probability $p_n = p_o + \Delta p$ is utilized, where the action $(\pm\Delta p)$ results in either an increment or a decrement. In case of $(+\Delta p)$, the ⊕ action has occurred, whereas $(-\Delta p)$ implies the ⊖ action. The change induced by one of these actions may be treated as two successive effects. First, the probability p_o is replaced by 0 and the replacement is recalculated. Second, the transition probability is subsequently reestablished with the new value p_n and the change in the fundamental matrix is recalculated immediately after the first recalculation.

The relationships accounting for the changes in the fundamental matrix $\overline{P}_T$ as a result of the element α having a null value and affecting the matrix P_T, as well as the relationships accounting for the changes in $\overline{P}_T$, also in the reverse case of ⊕ actions, may be found in Appendix I.

In sum, for the search of extreme subsystems following the theory of constructing the defining sequences on system W elements with the aid of KSR routines introduced in the previous work (Mullat, 1976), it is necessary to obtain some well-organized and distinct recurrent expressions, which can account for the changes in the matrix $\overline{P}_T$ whereby it is transformed to the matrix $\overline{P}_{T\cup\alpha}$. The formulas for specified Δp, which allow us to transform from $\overline{P}_T$ in order to find the matrix $\overline{P}_{T\cup\alpha}$ are given in Appendix II on the basis of the expressions II 1.3 and II 1.4.

With the aid of these recurrent expressions, in Appendix II, it is possible to obtain on each set $H \subseteq W$ the collection of credentials $\Pi^{+}H$ or $\Pi^{-}H$ by performing the successive implementation of expressions II 2.5 to all elements upon the set $\overline{H}$. These expressions mirror the transformation of system element credentials π into π_{α} in view of the theoretical apparatus of monotonic systems (Mullat, 1976). Indeed, we construct the collection $\Pi^{+}H$ in the case of $\Delta p > 0$, whereas the collection $\Pi^{-}H$ is constructed if $\Delta p < 0$.

4. On homogeneous Markov chains

In this section, we consider homogeneous Markov chains with a finite number n of states and a discrete time. A chain is called homogeneous if and only if the transition probabilities $p_{i,j}$ are independent of time t.

Our goal is to establish the relations between the elements of fundamental matrix denoting an absorbing chain (Chung, 1960), p. 66), see the definition below on the condition that certain transitions per time unit have been declared as prohibited. These relations are used in adjusting the corresponding elements without imposing this restriction. It should be noted that similar relations are encountered in compositions pertaining to the first and the last occurrence of some Markov chain states (see (Chung, 1960), p. 75). However, in spite of this obvious resemblance, such relations have not yet been considered in the literature.

Given without proof, the relations given in the form of theorems I-IV allow making a case for implementation of a general principle of maximum for some functions, defined on finite sets (Mullat, 1971). The foundation for the construction scheme (1971), in particular, is contingent upon requirements applied to the functions in the form of inequalities given as a result of this research.

In developing an efficient algorithm at the computer center of the Tallinn University of Technology, the theorems I-IV served as a foundation for finding solutions for some notable pattern recognition classification problems. Application of the algorithm improved the solution quality and speed with which problems were solved computationally, in comparison with those achieved by currently used algorithms.

Usually, homogenous chain can be represented as a directed graph whose vertices correspond to the state of the chain, whereby the arcs denote possible unit transitions from one state to another at any point in time. In addition, when the transition probability $p_{i,j}$ is zero, the arc $u = (i, j)$ is not depicted on the graph. On the other hand, any graph Γ can be represented in the form of a homogeneous chain attributing the arcs of the chain by satisfying the relation of the conditional probabilities. These chains are referred to as chains associated with the graph Γ.

Let $U(G)$ be the set of arcs of the graph G, and $V(G)$ the set of vertices. Adding to the set of vertices $V(G)$ a vertex θ, which is in turn connected to any vertex in $V(G)$ by an arc leading into θ, can hence reproduce a graph Γ

Consider the following homogeneous Markov chain associated with the graph G:

1) There exists a unique absorbing state $\theta \notin V(G)$;

2) The probability of transition from i to j, $i, j \in V(G)$, $p_{i,j} = p_j$, if the arc $(i, j) \in U(G)$, and $p_{i,j} = 0$ otherwise;

3) The probability of transition from the state $i \in V(G)$ to the absorbing state θ is given by $p_{i,\theta} = 1 - \sum_{i=1}^{n} p_{i,j}$.

It can easily be verified that all states of the chain, identified by the vertices of the graph G, are irrevocable, whereby the designated Markov chain belongs to a class of absorbing chains (see (Chung, 1960), p. 55).

Here, some of the tuning indicators v_j refer to the parameters of the Markov chain associated with the graph G. Further, we assume that for any $v_j = \sum_i^n p_{i,j} < 1$. For all vertices of the graph G, it can be demonstrated that for any graph G, one can find a tuning parameter v for which a given constraint $0 < v < 1/k$ is satisfied. Indeed, let k represent the largest number of nonzero elements in the rows of the fundamental matrix corresponding to the vertices of the graph G.

Moreover, let H denote an arbitrary subset of arcs of the graph G, i.e., $H \subset U(G)$. Here, $p(H, i, j, k)$ designates the probability of transition from the state i to the state j in k units of time, on the condition that the transitions along the arcs of the subset H are prohibited during this period. Owing to this restriction, the subset H denotes a prohibited set of arcs, all of which are thus prohibited as well.

Let $p(H, i, j, 0) = \delta_{i,j}$ (where $\delta_{i,j}$ represents the Kronecker's symbol) and

$$\bar{p}(H, i, j) = \sum_{n=0}^{\infty} p(H, i, j, n).$$

Due to the existence of a Markov chain associated with the graph Γ of an absorbing state θ, the entire set $V(G)$ is irrevocable, see Chung, 1960, p. 45, and the series (1) converges.

We use the Greek letters $\alpha, \beta, \ldots$ to denote prohibited arcs of the graph G, whereby α^{+} refers to the vertex (state) from which the arc emerges, and α^{-} is the vertex toward which the arc is pointing.

Theorem I. We denote by $H + \alpha$ a set-theoretic operation. $H \cup \alpha$.

$$\bar{p}(H+\alpha, i, j) = \bar{p}(H, i, j) -$$
$$- v \cdot \frac{\bar{p}(H, i, \alpha^{+}) \cdot \bar{p}(H, \alpha^{-}, j)}{1 + p_{\alpha^{-}} \cdot \bar{p}(H, \alpha^{-}, \alpha^{+})} \cdot$$

This expression might be interpreted as a consequence of malfunctions in the communication line α.

Theorem II. This can be interpreted as an increase in traffic efficiency after repairs on the line.

$$\bar{p}(H, i, j) = \bar{p}(H+\alpha, i, j) +$$
$$+ v \cdot \frac{\bar{p}(H+\alpha, i, \alpha^{+}) \cdot \bar{p}(H+\alpha, \alpha^{-}, j)}{1 - p_{\alpha^{-}} \cdot \bar{p}(H, \alpha^{-}, \alpha^{+})}$$

Corollary.

From the form of the dependence in the formulations of Theorems I-II it immediately follows that the following inequalities are valid for the case of directed and undirected graphs, respectively

$$\bar{p}(H+\alpha, i, j) \le \bar{p}(H, i, j),\ i, j = \overline{1, n}).$$

These inequalities guarantee the fulfillment of the monotonicity condition for the realization of a monotonic system on homogeneous Markov chains.

APPENDIX I

Consider the value $\bar{p}(T, i, j)$ produced by the series (3). Each component of this series may be treated as the measure of all passes in m time steps (time units) commencing in i and terminating in j. This assemblage of transitions is a union of two nonintersecting collections. The first set pertains to the passes from i to j with a mandatory transition, at least once, along $\alpha \in W$. On the other hand, the second relates to the set of passes from i to j avoiding this transition α. Each passage from the first set consists of two passes: a pass avoiding α being in t steps long, and a pass in $m-t-1$ steps (time units), passing along α. In other words, the passages in t steps avoid the pass along α, whereas passages in $m-t-1$ steps make use of this pass α.

We introduce the following notation: $\overline{p}(T^0, i, j, k)$ represents the average number of hits from i into j with the transition matrix P_T, where the nonzero element α is null, and $p(T^0, i, j, k)$ denotes the probability of transition without making use of α. Implementation of the introduced notification results in:

$$p(T,i,j,m) = p(T^0,i,j,m) + p_\alpha \cdot \sum_{t=0}^{m-1} p(T^0,i,\alpha_b,t) \cdot p(T,\alpha_e,j,m-t-1); \qquad \text{II}$$

$$p(T,i,j,m) = p(T^0,i,j,m) + p_\alpha \cdot \sum_{t=0}^{m-1} p(T,i,\alpha_b,t) \cdot p(T^0,\alpha_e,j,m-t-1), \qquad \text{II}$$

where α_b – the state from which a one-step pass begins, ending in α_e; p_α – the pass along α in one step, corresponding to the element α of the matrix P_T.

The first component in II 1.1 and II 1.2 introduces the value of $p(T,i,j,m)$, denoting the measure of transitions avoiding the pass along α. In addition, the components included in the summation represent the probability that the states α_b (for the relationship II 1.1) and α_e (for the relationship II 1.2) have been reached by the first and the last pass along α in the moments t and $t+1$, respectively.

Let us calculate the $\overline{p}(T,i,j)$ values using the relationship II 1.1. We conclude, after performing the summation of each of the equations II 1.1 from 1 to M and thereafter changing the order of sums in the double summation, that

$$\sum_{m=1}^{M} p(T,i,j,m) = \sum_{m=1}^{M} p(T^0,i,j,m) \; p_\alpha \cdot \sum_{t=0}^{M-1} p(T^0,i,\alpha_b,t) \cdot \sum_{s=1}^{M-t} p(T,\alpha_e,j,s-1).$$

Dividing both parts of the latter equation yields $\sum_{t=0}^{M-1} p(T^0,i,\alpha_b,t)$.

Thus, based on the theorem of Norlund averages (Chung 1960) considering the sequence $a_t = p(T^0, i, \alpha_b, t)$ and $b_{m-t} = \sum_{s=1}^{M-t} p(T, \alpha_e, j, s-1)$, while increasing $M \to \infty$ for the sequences a_n and b_n, it can be concluded that the following relations are valid:

$$\overline{p}(T,i,j) = \overline{p}(T^0,i,j) + p_\alpha \cdot \overline{p}(T^0,i,\alpha_b) \cdot \overline{p}(T,\alpha_e,j). \quad \text{II 1.3}$$

Analogous relationship can be deduced by exploiting the composition II 1.2, namely:

$$\overline{p}(T,i,j) = \overline{p}(T^0,i,j) + p_\alpha \cdot \overline{p}(T,i,\alpha_b) \cdot \overline{p}(T^0,\alpha_e,j). \quad \text{II 1.4}$$

APPENDIX II

We introduce the following notifications. Let $\overline{p}(T_o, i, j)$ represent the matrix $\overline{P}_T$ element, and $\overline{p}(T_n, i, j)$ denote the matrix $\overline{P}_{T\cup\alpha}$ element. Let us also rewrite II 1.3 and II 1.4 with respect to these notifications, which results in:

$$\overline{p}(T_n,i,j) = \overline{p}(T^0,i,j) + p_n \cdot \overline{p}(T^0,i,\alpha_b) \cdot \overline{p}(T_n,\alpha_e,j); \quad \text{II 2.1}$$

$$\overline{p}(T_o,i,j) = \overline{p}(T^0,i,j) + p_o \cdot \overline{p}(T_o,i,\alpha_b) \cdot \overline{p}(T^0,\alpha_e,j). \quad \text{II 2.2}$$

From the relationships II 2.1 and II 2.2, it follows that the new value for the average hits from i into j is equal to

$$\begin{aligned}\overline{p}(T_n,i,j) &= \overline{p}(T_o,i,j) + p_n \cdot \overline{p}(T^0,i,\alpha_b) \cdot \overline{p}(T_n,\alpha_e,j) - \\ &- p_o \cdot \overline{p}(T_o,i,\alpha_b) \cdot \overline{p}(T^0,\alpha_e,j)\end{aligned}. \quad \text{II 2.3}$$

Substituting in II 2.1 the state $i = \alpha_e$, we obtain

$$\overline{p}(T_n,\alpha_e,j) = \overline{p}(T^0,\alpha_e,j) \big/ \left(1 - p_n \cdot \overline{p}(T^0,\alpha_e,\alpha_b)\right),$$

and from II 2.2, with the same $i = \alpha_e$, we get

$$\overline{p}(T^0,\alpha_e,j) = \overline{p}(T_o,\alpha_e,j) \big/ \left(1 + p_o \cdot \overline{p}(T_o,\alpha_e,\alpha_b)\right).$$

Replacing the latter expression into the preceding one, and taking into account that

$$\overline{p}(T^0,\alpha_e,\alpha_b) = \overline{p}(T_o,\alpha_e,\alpha_b) \big/ \left(1 + p_o \cdot \overline{p}(T_o,\alpha_e,\alpha_b)\right),$$

we finally arrive at

$$\overline{p}(T_n,\alpha_e,j) = \overline{p}(T_o,\alpha_e,j) \big/ \left(1 - \Delta p \cdot \overline{p}(T_o,\alpha_e,\alpha_b)\right). \quad \text{II 2.4}$$

The expression II 2.1 is valid if we replace T_n by T_o and p_n by p_o, and if in the expression II 2.2 we make a reverse replacement. Substituting $j = \alpha_n$ in the expression II 2.2, first regrouping it by this reverse replacement, results in

$$\overline{p}(T^0, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right).$$

Finally, we deduce the expression that can account for the changes in the fundamental matrix $\overline{P}_T$ by simplifying the last two equalities and the expression II 2.4, after collecting sub-expressions and making rearrangements to transform $\overline{P}_T$ into the matrix $\overline{P}_{T\cup\alpha}$. Adopting the standard nomenclature given in Section III, the ultimate form of the expression is given as follows:

$$\overline{p}(T \cup \alpha, i, j) = \overline{p}(T, i, j) + \Delta p \cdot \frac{\overline{p}(T, i, \alpha_b) \cdot \overline{p}(T, \alpha_k, j)}{1 - \Delta p \cdot \overline{p}(T, \alpha_e, \alpha_b)}. \quad \text{II 2.5}$$

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