

## ЭКСТРЕМАЛЬНЫЕ ПОДСИСТЕМЫ МОНОТОННЫХ СИСТЕМ. II

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Предлагается конструктивная процедура построения особых определяющих последовательностей элементов монотонных систем, рассмотренных в [1]. Изучаются взаимные свойства двух определяющих последовательностей  $\bar{\alpha}_-$  и  $\bar{\alpha}_+$ , и полученный результат формулируется в виде теоремы двойственности. На основе теоремы двойственности описан способ сужения области поиска экстремальных подсистем — ядер монотонной системы и приведена соответствующая схема поиска.

### 1. Введение

В [1] разработан основной аппарат выделения в монотонных системах особых подсистем — ядер, обладающих экстремальными свойствами. Основным понятием развитого аппарата является определимое множество [2]. В принятой терминологии определимое множество оказывается наибольшим ядром монотонной системы взаимосвязанных элементов. Понятие определимого множества в [1] вводилось с помощью предположения существования особых подпоследовательностей элементов изучаемой системы, названных определяющими ( $\bar{\alpha}_-$  и  $\bar{\alpha}_+$ ) — последовательностями.

В данной работе вопрос существования определяющих последовательностей решается конструктивно в виде процедур — алгоритмов. Основные свойства определяющей последовательности, построенной по правилам процедуры и исчерпывающей все множество элементов системы  $\mathcal{W}$ , гарантируется теоремой.

Рассматривается также вопрос о том, какая существует связь между определяющими последовательностями  $\bar{\alpha}_-$  и  $\bar{\alpha}_+$ . Можно предположить, что если построена определяющая последовательность  $\bar{\alpha}_-$ , то стоит взять эту последовательность в обратном порядке, как получится  $\bar{\alpha}_+$  последовательность. В общем случае это не так. Тем не менее имеет место более слабое утверждение. На основе определенных в [1] понятий дискретных действий типа  $\oplus$  и  $\ominus$  и на элементы системы  $\mathcal{W}$  данное утверждение формулируется здесь в виде теоремы двойственности. В случае выполнения условий теоремы двойственности изложенные алгоритмы построения определяющих последовательностей используются для значительного сужения области поиска  $\oplus$  и  $\ominus$  ядер системы  $\mathcal{W}$ . Алгоритм сужения области поиска изложен также в виде процедуры — конструктивно.

## Extremal Subsystems of Monotonic Systems, II

**Abstract.** A constructive routine is considered for obtaining singular defining sequence of elements of monotonic systems studied by Mulla (1976). The relationship between two defining sequences  $\overline{\alpha}_-$  and  $\overline{\alpha}_+$  is also examined, and the obtained result is formulated as a duality theorem. This theorem is used for describing a routine of restricting the domain of search for extremal subsystems (or kernels of a monotonic system); the corresponding search scheme is also presented.

Keywords: monotonic; system; matrix; graph; cluster

### 1. INTRODUCTION

In Mulla (1976) we have developed the basic method of selection (from monotonic systems) of singular subsystem, i.e., the kernels possessing extremal properties. The main concept of this method is that of a definable set Mulla (1971). In the terminology adopted by us, a definable set is the largest kernel of a monotonic system of interrelated elements. In 1971 we introduced the concept of a definable set with the aid of the system under consideration called defining  $\overline{\alpha}_-$  ( $\overline{\alpha}_+$ ) sequences.

In this paper the problem of existing of defining sequences is solved constructively in the form of routines (algorithms). The principal properties of defining sequences sequence constructed according to the rules of a routine and that exhausts the entire set of elements of the system  $\mathbf{W}$  are specified by a theorem.

We shall also examine the relationship between two defining sequences  $\overline{\alpha}_-$  and  $\overline{\alpha}_+$ . It can be assumed that after constructing a defining sequence  $\overline{\alpha}_-$ , we could take this sequence in inverse order, thus obtaining an  $\overline{\alpha}_+$  sequence. But in the general case this is not so. Nevertheless we can make a weaker assertion. On the basis of the concepts (defined in Mulla (1976) of discrete operations of type  $\oplus$  and  $\ominus$  on the elements of a system  $\mathbf{W}$ , this assertion will be formulated below as a duality theorem. Under the conditions of the duality theorem, the algorithms of construction of defining sequences described here will be used for considerably restricting the domain of search for  $\oplus$  and  $\ominus$  kernels of the system  $\mathbf{W}$ . The algorithm of restriction of the domain of search is presented in the form of a constructive routine.

### 2. ROUTINE OF FINDING THE KERNELS

Below we describe a routine of construction of an ordered sequence  $\overline{\alpha}$  of all the elements of  $\mathbf{W}$ . In abbreviated form, this routine is called KSR (kernel-searching routine).

This routine consists of rules of generation and scanning of an ordered series of ordered sets  $\langle \bar{\beta}_j \rangle$  (sequences); here  $j$  varies from zero to a value  $p$ , which is automatically determined by the rules of the routine, whereas the elements of each sequence  $\bar{\beta}_j$  are selected from the set  $W$ <sup>1</sup>.

This series  $\langle \bar{\beta}_j \rangle$  constructed by this rule forms a numerical sequence of thresholds  $\langle u_j \rangle$  and a sequence of sets  $\langle \Gamma_j \rangle$ . On the other hand the sequence of thresholds governs the transactions from  $\bar{\beta}_{j-1}$  to  $\bar{\beta}_j$  in the chain  $\langle \bar{\beta}_j \rangle$ , and the sequence  $\langle \Gamma_j \rangle$  terminates with a set, which is definable.

In the description of a rule we use the operation of extending a sequence  $\bar{\beta}_j$  by adjoining to it another sequence  $\bar{\gamma}$ . This operation is symbolically expressed by  $\bar{\beta} \leftarrow \langle \bar{\beta}, \bar{\gamma} \rangle$ . This rule of construction of the sequence  $\bar{\alpha}$  of all elements of the set  $W$  can be described stages: by step **Z** and **R**.

**Z.** In the set  $W$  we find an element  $\mu_0$  such that  $\pi^- W(\mu_0) = \min_{\delta \in W} \pi^- W(\delta) = F_-(W)$  we are constructing a defining sequence  $\bar{\alpha}_-$ . The construction of  $\bar{\alpha}_+$  is entirely similar and therefore not presented here. We shall only indicate where it is necessary to invert the sign of inequalities, and where the search for an element with the minimal credential must be replaced by search for an element with maximal credential, so as to be able to construct  $\bar{\alpha}_+$ . Thus the construction here of  $\bar{\alpha}_+$ , the element  $\mu_0$  is obtained from the condition  $\pi^+ W(\mu_0) = \max_{\delta \in W} \pi^+ W(\delta) = F_+(W)$ . We shall write  $u_0 = \pi^- W(\mu_0)$ ,  $\bar{\alpha} = \langle \mu_0 \rangle$  and the set  $\Gamma_0 = W$ . We select a subset of elements  $\gamma$  from  $W$  such that  $\pi^- W \setminus \bar{\alpha}(\gamma) \leq u_0$ . The construction of  $\bar{\alpha}_+$  requires the selection of such  $\gamma$  that  $\pi^+ W \setminus \bar{\alpha}(\gamma) \geq u_0$ ,  $u_0 = \pi^+ W(\mu_0)$ . After that we order the elements in a certain manner (which can be arbitrary selected). The thus-obtained ordered set is denoted by  $\bar{\gamma}$ . Let us write  $\bar{\beta}_0 = \bar{\gamma}$ .

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<sup>1</sup> Let us recall that in a) the brackets  $\langle \rangle$  denoted an ordered set; in the case under consideration they denote an ordered set of ordered sets  $\bar{\beta}_j$ .

R. We construct a recursive routine for extending the sequences  $\bar{\alpha}$  and  $\bar{\beta}_0$ . Here we denote by  $\beta_0(i)$  the  $i$ -th element of the sequence  $\bar{\beta}_0$ . We specify one after another the elements of the sequence  $\bar{\beta}_0$ . At each instant of specification we extend the sequence  $\bar{\alpha}$  by the elements from  $\bar{\beta}_0$  of the sequence fixed at this instant. In accordance with the symbolic notation of the operation of extension of a sequence  $\bar{\alpha}$ , we perform at each instant  $t$  of specification the operation  $\bar{\alpha} \leftarrow \langle \bar{\alpha}, \beta_0(t) \rangle$ . Suppose that all the elements of  $\bar{\beta}_0$  up to  $\beta_0(i-1)$  inclusive have been fixed. Then the sequence  $\bar{\alpha}$  will have the form  $\langle \mu_0, \beta_0(1), \beta_0(2), \dots, \beta_0(i-1) \rangle$ , which corresponds to the symbolic notation of the operation of extension of the sequences  $\bar{\alpha} \leftarrow \langle \bar{\alpha}, \beta_0(1), \beta_0(2), \dots, \beta_0(i-1) \rangle$  in the case that  $\bar{\alpha}$  inside the brackets consists of one element  $\mu_0$ . Let us consider an element  $\beta_0(i-1)$  of the sequence  $\bar{\beta}_0$ . At the instant of specification of the element  $\beta_0(i-1)$  we decide during the above-mentioned operation of extension of  $\bar{\alpha}$  also about any further extension or about stopping the extension of the sequence  $\bar{\beta}_0$ . We must check the following two conditions:

- a) In the set  $W \setminus \bar{\alpha}$  there exist elements such that  $\pi^- W \setminus \bar{\alpha}(\gamma) \leq u_0$ .  
In constructing  $\bar{\alpha}_+$ , this condition is replaced by  $\pi^+ W \setminus \bar{\alpha}(\gamma) \geq u_0$ ;
- b) the element  $\beta_0(i)$  is defined for the sequence  $\bar{\beta}_0$ . By assumption an element  $\beta_0(i)$  to be defined for a sequence  $\bar{\beta}_0$  if the sequence  $\bar{\beta}_0$  has an element with an ordinal number  $i$ . Otherwise the element  $\beta_0(i)$  is not defined. There can be four cases of fulfillment or no fulfillment of these conditions. In two cases, when the first condition is satisfied, irrespective of whether or not the second condition holds, the sequence  $\bar{\beta}_0$  will be extended. This means that the set of elements  $\gamma$  in  $W \setminus \bar{\alpha}$  specified by the first condition is ordered in the form of sequence  $\bar{\gamma}$ . The sequence  $\bar{\beta}_0$  is extended in accordance with the formula  $\bar{\beta}_0 \leftarrow \langle \bar{\beta}_0, \bar{\gamma} \rangle$ . In case when the first condition is not satisfied, whereas the second condition is satisfied, we shall fix the element  $\beta_0(i)$  and at the same time extend the sequence  $\bar{\alpha}$ , i.e.,

$\bar{\alpha} \leftarrow \langle \bar{\alpha}, \beta_0(i) \rangle$ , and proceed to new recursion stage. In case neither the first nor the second condition holds, the sequence  $\bar{\beta}_0$  will not be extended nor the last fixed element in the sequence  $\bar{\beta}_0$  will be the element  $\beta_0(i-1)$ . Suppose that we have fixed all the elements of the sequence  $\bar{\beta}_j$ . By that time we have constructed a sequence  $\bar{\alpha}$ . Let us consider the set  $W \setminus \bar{\alpha}$  and the credential system  $\Pi^- W \setminus \bar{\alpha}$ . We shall find an element in  $\Pi^- W \setminus \bar{\alpha}$  on which the minimum is reached in the credential system  $\Pi^- W \setminus \bar{\alpha}$ . The obtained element is denoted by  $\mu_{j+1}$ . We obtain  $\bar{\alpha}_+$  the element  $\mu_{j+1}$  from the condition:  $\pi^+ W \setminus \bar{\alpha}(\mu_{j+1}) = \max_{\delta \in W \setminus \bar{\alpha}} \pi^+ W \setminus \bar{\alpha}(\delta) = F_+(W \setminus \bar{\alpha})$ . Thus,  $\pi^- W \setminus \bar{\alpha}(\mu_{j+1}) = F_-(W \setminus \bar{\alpha})$ . Let us write  $u_{j+1} = \pi^- W \setminus \bar{\alpha}(\mu_{j+1})$ , and for the set  $\Gamma_{j+1} = W \setminus \bar{\alpha}$ ; then we supplement the sequence  $\bar{\alpha}$  by the element  $\mu_{j+1}$ , i.e.,  $\bar{\alpha} \leftarrow \langle \bar{\alpha}, \mu_{j+1} \rangle$ . In the same way as during the zero step, we select a subset of elements  $\gamma$  from  $W \setminus \bar{\alpha}$  such that  $\pi^- W \setminus \bar{\alpha}(\gamma) \leq u_{j+1}$ . Here we select for  $\bar{\alpha}_+$  a set of elements  $\gamma$  such that  $\pi^+ W \setminus \bar{\alpha}(\gamma) \geq u_{j+1}$ . The selected set can be ordered in any manner. The ordered set is denoted by  $\bar{\gamma}$ . The set  $\bar{\beta}_{j+1}$  is assumed to be equal to  $\bar{\gamma}$ .

- c) By analogy with previous b) the recursion step will be described as a recursion routine. At this stage we also use the rule of extension of the sequences  $\bar{\alpha}$  and  $\bar{\beta}_{j+1}$ . Suppose that we have fixed all elements of  $\bar{\beta}_{j+1}$  up to  $\beta_j(i-1)$  inclusive. Then the sequence  $\bar{\alpha}$  will have the form  $\bar{\alpha} = \langle \bar{\alpha}, \mu_{j+1}, \beta_j(1), \dots, \beta_j(i-1) \rangle$ , where  $\bar{\alpha}$  denotes the sequence  $\bar{\alpha}$  obtained at the instant of fixing all the elements of  $\bar{\beta}_j$ , or, to rephrase, the sequence  $\bar{\alpha}$  prior to the  $(j+1)$ -th step. The last equation corresponds to the symbolic operation of extension of the sequence  $\bar{\alpha} = \langle \bar{\alpha}, \mu_{j+1}, \beta_j(1), \dots, \beta_j(i-1) \rangle$  in the case that  $\bar{\alpha}$  inside the

brackets denotes the sequence  $\langle \bar{\alpha}, \mu_{j+1} \rangle$ . Let us consider an element  $\beta_{j+1}(i-1)$  of the sequence  $\bar{\beta}_{j+1}$ . At the instant of fixing the element  $\beta_{j+1}(i-1)$  we decide about a further extension or about stopping the extension of the sequence  $\bar{\beta}_{j+1}$ . For this purpose we consider the credential system  $\Pi^-W \setminus \bar{\alpha}$  and we check two conditions:

- 1) The set  $W \setminus \bar{\alpha}$  contains elements  $\gamma$  such that  $\pi^-W \setminus \bar{\alpha}(\gamma) \leq u_{j+1}$ . For constructing  $\bar{\alpha}_+$  we must take elements  $\gamma$  such that  $\pi^+W \setminus \bar{\alpha}(\gamma) \geq u_{j+1}$ ;
- 2) the element  $\beta_{j+1}(i)$  is defined for the sequence  $\bar{\beta}_{j+1}$ .

By analogy with the step **Z**, we find that the sequence  $\bar{\beta}_{j+1}$  is extended in two cases in which the first condition is satisfied irrespective of whether or not the second condition holds. The set of elements  $\gamma$  in  $W \setminus \bar{\alpha}$  specified by the first condition is ordered in the form of a sequence  $\bar{\gamma}$ . The sequence  $\bar{\beta}_{j+1}$  is extended in accordance with the formula  $\bar{\beta}_{j+1} \leftarrow \langle \bar{\beta}_{j+1}, \bar{\gamma} \rangle$ . In the case that the first condition does not hold, whereas the second condition is satisfied, the element  $\beta_{j+1}(i)$  will be fixed and at the same time we extend the sequence  $\bar{\alpha}$ , i.e.,  $\bar{\alpha} \leftarrow \langle \bar{\alpha}, \beta_{j+1}(i) \rangle$ , and after that we proceed again in accordance with the rules of Stage 2 of the recursion routine of extension of the sequence  $\bar{\beta}_{j+1}$ . In the case that neither the first, nor the second condition holds, the sequence  $\bar{\beta}_{j+1}$  will not be extended, and the last fixed element of the sequence  $\bar{\beta}_{j+1}$  will be the element  $\beta_{j+1}(i-1)$ . At some step **p** the sequence  $\bar{\alpha}$  will exhaust the entire set of elements  $W$ .

**Theorem 1.** A sequence  $\bar{\alpha}$  constructed on the basis of a collection of credential system  $\{ \Pi^-H \mid H \subseteq W \}$  is a defining sequence  $\bar{\alpha}_-$ , whereas a sequence  $\bar{\alpha}$  constructed on the basis of  $\{ \Pi^+H \mid H \subseteq W \}$  is a defining sequence  $\bar{\alpha}_+$ .

The first part of the theorem (for  $\overline{\alpha}_-$ ) is proved in Appendix 1. The second part (for  $\overline{\alpha}_+$ ) can be proved in the same way.

NB1. Let us note that a sequence  $\overline{\alpha}$  constructed by KSR rules has somewhat stronger properties than required in obtaining a defining sequence. More precisely, there does not exist a proper subset  $L$  for  $j = 0, 1, \dots, p-1$  such that  $\Gamma_j \supset L \supset \Gamma_{j+1}$  and  $F_-(\Gamma_j) < F_-(L)$ . This is not required for obtaining a defining sequence  $\overline{\alpha}_-$  ( $\overline{\alpha}_+$ ). The corresponding proof is not given here.

NB2. Let us note another circumstance. With the aid of the kernel-searching routine it is possible to effectively find (without scanning) the largest kernel, i.e., a definable set. It is not possible to find an individual kernel strictly included in a definable set (if the latter exists) by constructing a defining sequence.

### 3. DUALITY THEOREM

Let us establish a relationship between the defining sequences  $\overline{\alpha}_-$  and  $\overline{\alpha}_+$  of a system  $W$ .

**Theorem 2.** Let  $\overline{\alpha}_-$  and  $\overline{\alpha}_+$  be defining sequences of the set  $W$  with respect to the collection of credential system  $\{\Pi^-H \mid H \subseteq W\}$ ,  $\{\Pi^+H \mid H \subseteq W\}$  respectively. Let  $\langle \Gamma_j^- \rangle$  be the subsequence of the sequence  $\Delta_{\overline{\alpha}_-}$  ( $j = 0, 1, \dots, p$ ) needed in the determination of  $\overline{\alpha}_-$ , and let  $\langle \Gamma_j^+ \rangle$  be the corresponding subsequence of the sequence  $\Delta_{\overline{\alpha}_+}$  ( $j = 0, 1, \dots, q$ ).

Hence if for an  $m$  and a  $n$  we have

$$F_+(\Gamma_n^+) = F_-(\Gamma_m^-), \quad (1)$$

then  $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$ ,  $\Gamma_n^+ \subseteq W \setminus \Gamma_{m+1}^-$ . If

$$F_+(\Gamma_n^+) < F_-(\Gamma_m^-)^2, \quad (2)$$

then  $\Gamma_m^- \subseteq W \setminus \Gamma_n^+$ ,  $\Gamma_n^+ \subseteq W \setminus \Gamma_m^-$ .

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<sup>2</sup> In the following, the + and - sign will not be used twice in notation. This rule applies also to Appendices 1 and 2

This theorem is important from two points of view. Firstly, under the conditions (1) and (2) there exists a relationship between an  $\overline{\alpha}_-$  sequence and  $\overline{\alpha}_+$ . This relationship consists in the fact that elements of  $\overline{\alpha}_+$  which are at the “beginning” and form either the set  $W \setminus \Gamma_{n+1}^+$  or the set  $W \setminus \Gamma_n^+$  will include all the elements of the set  $\Gamma_m^-$  that are at the “end” of  $\overline{\alpha}_-$ . The same applies also to sets  $W \setminus \Gamma_{m+1}^-$  or  $W \setminus \Gamma_m^-$  which are at the beginning of  $\overline{\alpha}_-$ , since they include in a similar way the set  $\Gamma_n^+$ . In other words, the theorem states that the sequence  $\overline{\alpha}_+$  does not differ “very much” (under certain conditions) from the sequence, which is the inverse to  $\overline{\alpha}_-$ .

Let us note that the conditions (1) and (2) are sufficient conditions, and it can happen that actual monotonic systems satisfying these conditions do not exist. Nevertheless, in the third part of this article, we shall describe actual examples of such systems.

#### 4. KERNEL SEARCH ROUTINE BASED ON DUALITY THEOREM

We just noted that a defining sequence  $\overline{\alpha}_+$  differs “slightly” from the inverse sequence of  $\overline{\alpha}_-$ . For elucidating the possibility of a search for kernels on the basis of the duality theorem, let us rephrase the latter. This assertion can be formulated as follows: at the beginning of the sequence  $\overline{\alpha}_+$  we often encounter elements of the sequence  $\overline{\alpha}_-$ , which are at the end of the latter.

Such an interpretation of the duality theorem yields an efficient routine of dual search for  $\oplus$  and  $\ominus$  kernels of the system  $W$ . This is due to the fact if the elements are often encountered, there exists a higher possibility of finding a  $\oplus$  kernel at the beginning of the sequence  $\overline{\alpha}_+$  as compared to finding it at the end of  $\overline{\alpha}_-$ ; the same applies also to a  $\ominus$  kernel in the sequence  $\overline{\alpha}_-$ .

The routine under construction is based on Corollaries I-IV of the duality theorem presented in Appendix II, where we also prove this theorem.

The routine of dual search for kernels described below is an application of two constructive routines, i.e., a KSR for constructing  $\overline{\alpha}_+$  and a KSR for constructing  $\overline{\alpha}_-$ . The routine is stepwise, with two constructing stages realized at each step, i.e., a stage in which the KSR is used for constructing  $\overline{\alpha}_+$  with  $\oplus$  operations, and a stage in which the same routine is used for constructing  $\overline{\alpha}_-$  with the aid of  $\ominus$  operations on the elements of the system.

- Z.** At first we store two numbers:  $u_0^+ = F_+(W)$  and  $u_0^- = 0$ . After that we perform precisely Stage 1 and 2 of the zero step of the KSR used for constructing the defining sequence  $\bar{\alpha}_+$ . This signifies that the set  $W$  contains an element  $\mu_0$  such that  $\pi^+ W(\mu_0) = \max_{\delta \in W} \pi^+ W(\delta) = F_+(W)$ . The threshold  $u_0^+$  is equal to  $\pi^+ W(\mu_0)$ , etc. By using the constructions of the zero step of KSR at the previous stage of the dual routine under construction, we obtained a set  $\Gamma_1^+ \subset W$ . Then we examine the set  $W \setminus \Gamma_1^+$  and the credential system  $\Pi^+ W \setminus \Gamma_1^+$ . On the set  $W \setminus \Gamma_1^+$  with the credential system  $\Pi^+ W \setminus \Gamma_1^+$  we perform a complete kernel-searching routine for the purpose of constructing a defining sequence of  $\oplus$  operations only for the set  $W \setminus \Gamma_1^+$ . As a result, we obtain in the set  $W \setminus \Gamma_1^+$  a subset  $K^1$  on which the function  $F_-$  reaches a global maximum among all the subsets of the set  $W \setminus \Gamma_1^+$ .
- R.** By applying the previous  $(j-1)$  steps to the  $j$ -th step, we obtained a sequence of sets  $\Gamma_0^+, \Gamma_1^+, \dots, \Gamma_j^+$ , and according to the construction of a defining sequence we have  $\Gamma_0^+ \supset \Gamma_1^+ \supset \dots \supset \Gamma_j^+$  and  $\Gamma_0^+ = W$ . At first we store two numbers:  $u_j^+ = F_+(\Gamma_j^+)$  and  $u_j^- = F_-(H^j)$ . By analogy, we perform the same construction consisting of two stages of a KSR recursion step for constructing  $\bar{\alpha}_+$  with the aid of  $\oplus$  operations. At a given instant of such dual construction we obtained a set  $\Gamma_{j+1}^+ \subset \Gamma_j^+$ . Then we consider the set  $W \setminus \Gamma_{j+1}^+$  and the credential system  $\Pi^- W \setminus \Gamma_{j+1}^+$ . In the same way as at the zero step, we perform on the set  $W \setminus \Gamma_{j+1}^+$  a complete kernel-searching routine with the purpose of constructing a sequence  $\bar{\alpha}_-$  only on the set  $W \setminus \Gamma_{j+1}^+$ . As a result we obtain in the set  $W \setminus \Gamma_{j+1}^+$  a subset  $H^{j+1}$  on which the function  $F_-$  reaches a global maximum among all subsets of the set  $W \setminus \Gamma_{j+1}^+$ . Before starting the construction of the  $j$ -th step of the routine under construction, we check the condition of a Rule of Termination of Construction Routine:

$$u_j^+ \leq u_j^- . \tag{3}$$

If (3) is satisfied as a strict inequality, the construction will terminate before the  $\dot{j}$ -th step. If (3) is an equality, the construction will terminate after the  $\dot{j}$ -th step.

### 5. DEFINABLE SETS OF DUAL KERNEL-SEARCH ROUTINE

At the end of the construction process, the above routine yields a set  $H^j$  or a set  $H^{j+1}$ . It can be asserted that one of the sets is definable set or the largest kernel of the system  $W$  with respect to a collection of credential system  $\{\Pi^-H \mid H \subseteq W\}$ .

The assertion is based on the following. Firstly, by applying the KSR we obtained the second stage of the  $\dot{j}$ -th step of a dual routine the maximal set  $H^{j+1}$  among all the subsets of the set  $W \setminus \Gamma_{j+1}^+$  on which the function  $F_-$  reaches a global maximum in the system of sets of all the subsets of the set  $W \setminus \Gamma_{j+1}^+$ . Secondly, by virtue of Corollary 1 of the Theorem 2 (the duality theorem), it follows that, prior to the  $\dot{j}$ -th step and provided that (3) is a strict inequality, the largest kernel (a definable set) will be contained in the set  $W \setminus \Gamma_j^+$ , or it follows from the Corollary 2 of the Theorem 2, if (3) is a equality, that the largest kernel is included in the set  $W \setminus \Gamma_{j+1}^+$ .

Thus by comparing these two remarks we can see that either  $H^j$  or  $H^{j+1}$  is a definable set.

By virtue of Corollaries 3 and 4 of the duality theorem, it is possible to find by similar dual routine also the largest kernel  $K^\oplus$ -definable set. This assertion can be proved in the same way as the assertion about  $H^j$  and  $H^{j+1}$ ; therefore this proof is not given here.

### APPENDIX 1

**Proof of Theorem 1.** We shell prove that a sequence  $\bar{\alpha}$  constructed by the KSR rules is a defining sequence for a collection of credential systems

$$\{\Pi^-H \mid H \subseteq W\}.$$

First of all let us recall the definition of a defining sequence of elements of the system  $W$ . We shall use the notation  $\Delta_{\bar{\alpha}} = \langle H_0, H_1, \dots, H_{k-1} \rangle$ , where  $H_0 = W$ ,  $H_{i+1} = H_i \setminus \alpha_i$  ( $i = 0, 1, \dots, k-2$ ). A sequence of elements of a set  $W$  is said to be defining with respect to a coalition of credential sys-

tem  $\{ \Pi^-H \mid H \subseteq W \}$  if the sequence  $\Delta_{\bar{\alpha}}$  has a subsequence of sets  $\Gamma_{\bar{\alpha}} = \langle \Gamma_0, \Gamma_1, \dots, \Gamma_p \rangle$ , such that

- a) The credential  $\pi^-H_i(\alpha_i)$  of any element  $\alpha_i$  of the sequence  $\bar{\alpha}$  that belongs to the set  $\Gamma_j$ , but does not belong to the set  $\Gamma_{j+1}$ , is strictly smaller than the credential of an element with minimal credential with respect to the set  $\Gamma_{j+1}$ , i.e.,  $\pi^-H_i(\alpha_i) < F_-(\Gamma_{j+1})$ ,  $j = 0, 1, \dots, p - 1$ <sup>3</sup>;
- b) the set  $\Gamma_p$  does not have a proper subset  $L$  such that the strict inequality  $F_-(\Gamma_p) < F_-(L)$  is satisfied (the “-” symbol has been omitted; see previous footnote).

We shall consider a sequence of sets  $\Delta_{\bar{\alpha}}$  and take the subsequence  $\Gamma_{\bar{\alpha}}$  in the form of the sets  $\Gamma_j$  ( $j = 0, 1, \dots, p$ ) constructed by the KSR rules. We have to prove that sets  $\Gamma_j$  have the required properties of a defining sequence. Assuming the contrary carries out the proof.

Let us assume that Mullat property (1971) of a defining sequence is not satisfied. This means that for any set  $\Gamma_j$  there exists in the sequence of elements

$$\bar{\beta}_j = \langle \beta_j(1), \beta_j(2), \dots \rangle$$

an element  $\beta_j(r)$  such that

$$\pi^-H_{v+r}(\beta_j(r)) \geq F_-(\Gamma_{j+1}) = u_{j+1} \tag{A.1}$$

Here  $v$  is the index number of the element  $\mu_j$  selected in Stage 1 of the recursion step of the constructive routine of determination of  $\bar{\alpha}$ ; in the vocabulary of notation used in Mullat (1976) we have  $v = i(\Gamma_j)$ .

According to the method of construction, the sequence  $\bar{\beta}_j$  consists of sequences  $\gamma$  formed at the second stage of the  $j$ -th step of the constructive rou-

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<sup>3</sup> In the definition of  $\bar{\alpha}_+$  sequence it is required that the following strict inequality be satisfied:  $\pi^+H_i(\alpha_i) > F_+(\Gamma_{j+1})$ ,  $j = 0, 1, \dots, q - 1$

time. Let  $\mathbf{M}$  be a set in a sequence of sets  $\Delta_{\bar{\alpha}}$  such that the first element  $\alpha_{i(\mathbf{M})}$  of the set  $\mathbf{M}$  in the constructed sequence  $\bar{\alpha}$  is used at the second stage of the  $j$ -th step for constructing the sequence  $\gamma$  to which the element  $\beta_j(\mathbf{r})$  belongs. This definition of  $\mathbf{M}$  shows that  $H_{v+r} \subseteq \mathbf{M}$ .

From the construction of the second stage of the  $j$ -th step and the principal property of monotonicity of  $\ominus$  operations in the system we obtain the inequalities

$$\pi^- H_{v+r}(\beta_j(\mathbf{r})) \leq \pi^- \mathbf{M}(\beta_j(\mathbf{r})) \leq \pi^- \Gamma_j(\mu_j) = u_j \quad (\text{A.2})$$

By virtue of the above method of selection of the set  $\Gamma_{j+1}$  from the sequence of sets  $\langle \Gamma_j \rangle$  and of the properties of a fixed sequence  $\bar{\beta}_j$ , we obtain at the  $j$ -th step

$$u_j = \pi^- \Gamma_j(\mu_{j+1}) < \pi^- \Gamma_{j+1}(\mu_{j+1}) = u_{j+1}, \quad (\text{A.3})$$

where  $j = 0, 1, \dots, p-1$ .

According to the rule of constructing of the sequence  $\bar{\alpha}$ , the function  $F_-$  reaches its value on the elements  $\mu_j$  and  $\mu_{j+1}$ . The elements  $\mu_j$  and  $\mu_{j+1}$  belong to the sets  $\Gamma_j$  and  $\Gamma_{j+1}$  respectively; therefore the inequalities (A.1) – (A.3) are contradictory.

Thus our assumption is not true and Mullat Property of the defining sequence  $\bar{\alpha}$  constructed by KSR rules has been proved.

Let us assume that Property b) does not hold, i.e., the last  $\Gamma_p$  of the sequence  $\langle \Gamma_j \rangle$  contains a proper subset  $L$  such that

$$F_-(\Gamma_p) < F_-(L). \quad (\text{A.4})$$

Let the element  $\lambda \in L$ , and suppose that it is the element with minimal ordinal number in  $\bar{\alpha}$  belonging to  $L$ ; moreover, let  $t$  denotes this number, i.e.,  $t = i(L)$ ,  $\alpha_t = \lambda$ . From the definition of  $t$  it follows that  $L \subseteq H_t$ .

Our analysis carried out above for the set  $H_{v+r}$  we repeat below for the set  $H_t$ . By analogy with the definition of the set  $\mathbf{M}$  we define a set  $\mathbf{M}'$  with the aid of the element  $\lambda$  and the sequence  $\bar{\alpha}$ .

The set  $M'$  is equated with the set of the sequence of sets  $\Delta_{\bar{\alpha}}$  that begins with an element used in the formation of a set  $\bar{\gamma}$  at the  $p$ -th step of the constructive routine such that  $\lambda \in \bar{\gamma}$ .

By analogy with derivative of (A.2) we obtain

$$\pi^-H_t(\lambda) \leq \pi^-M'(\lambda) \geq \pi^- \Gamma_p(\mu_p) = u_p. \quad (A.5)$$

Since  $F_-(L) \leq \pi^-L(\lambda)$ , it follows from (A.4) and (A.5) that  $\pi^-H_t(\lambda) < \pi^-L(\lambda)$ .

We noted above that  $L \subseteq H_t$ , by virtue of the monotonicity of  $\ominus$  operations, it hence follows that

$$\pi^-L(\lambda) \leq \pi^-H_t(\lambda).$$

The last two inequalities are contradictory, and hence Property b) of the defining sequence is satisfied.

Thus we have proved that the sequence  $\bar{\alpha}$  constructed by the KSR rules is a defining sequence with respect to a collection of credential systems  $\{\Pi^-H \mid H \subseteq W\}$ , and hence it can be denoted by  $\bar{\alpha}_-$ , whereas the sequence  $\langle \Gamma_j \rangle$  obtained by a constructive routine can be denoted by  $\Gamma_{\bar{\alpha}_-}$ .

**APPENDIX 2**

**Proof of Duality Theorem.** Below we shall show that  $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$ , if  $F_+(\Gamma_n^+) = F_-(\Gamma_m^-)$  (we omit a twice notation of + and - symbols; a promised above the + and - sign will not be used twice in notation. This rule have been applied also to Appendices 1 and 2.

Let us assume that there exists an element  $\xi \in \Gamma_m^-$  and that  $\xi \in \Gamma_{m+1}^-$ , i.e.,  $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$ . Hence follows that we have defined a credential  $\pi\Gamma_{n+1}^+(\xi)$ . According to the definition of the function  $F_+$  we have the inequality  $\pi\Gamma_{n+1}^+(\xi) \leq F(\Gamma_{n+1}^+)$ .

For a defining sequence  $\bar{\alpha}_+$  and for any  $j = 0, 1, \dots, q - 1$  we have inequalities  $F(\Gamma_{n+1}^+) < F(\Gamma_n^+)$ .

Let us consider an element  $\mathbf{g} \in \Gamma_n^+$  with the smallest index number in  $\overline{\alpha}_+$ . It follows from the definition of  $\overline{\alpha}_+$  that

$$\pi\Gamma_n^+(\mathbf{g}) > F(\Gamma_{n+1}^+). \quad (\text{A.7})$$

The choice of element  $\mathbf{g}$  is convenient because it permits the use of Mullat Property of a defining sequence (see Appendix 1), i.e., in this case the set  $\Gamma_n^+$  is in the form of  $H_t = \Gamma_n^+$ . Since  $F(\Gamma_n^+) \geq \pi\Gamma_n^+(\mathbf{g})$ , we have proved (A.6).

Since  $\xi \in \Gamma_m^-$ , it follows that we have defined a credential  $\pi\Gamma_m^-(\xi)$ . We have the following chain of inequalities:

$$F(\Gamma_m^-) \leq \pi\Gamma_m^-(\xi) \leq \pi^-W(\xi) = \pi^+W(\xi) \leq \pi\Gamma_n^+(\xi).$$

Let us recall that for any element  $\delta$  of the system  $W$  under consideration, we have in a) the relation  $\pi^-W(\delta) = \pi^+W(\delta)$ . The first inequality follows from the definition of the function  $F_-$ , and the second inequality from the monotonicity of  $\ominus$  operations. The equality follows from the definition of the functions  $\pi^-$  and  $\pi^+$ , whereas the last inequality follows from the monotonicity of  $\ominus$  operations.

By virtue of (A.6) and of the conditions of the theorem, we have also the following chain of inequalities:

$$\pi\Gamma_{n+1}^+(\xi) \leq F(\Gamma_{n+1}^+) < F(\Gamma_n^+) = F(\Gamma_m^-).$$

By supplementing this chain by the previous chain of inequalities, we hence obtain  $\pi\Gamma_{n+1}^+(\xi) < \pi\Gamma_n^+(\xi)$ . Since  $\Gamma_{n+1}^+ \subset \Gamma_n^+$ , it follows from the monotonicity of  $\oplus$  operations that  $\pi\Gamma_{n+1}^+(\xi) < \pi\Gamma_{n+1}^+(\xi)$ . The logical step used for obtaining the last inequality is valid, and therefore the assumption that  $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$  is untrue.

In the same way we can prove the inclusion  $\Gamma_n^+ \subseteq W \setminus \Gamma_{m+1}^-$ . For this purpose it suffices to change the signs of the inequalities and (whenever necessary) to replace the set  $\Gamma_{n+1}^+$  by  $\Gamma_{n+1}^-$ , and  $\Gamma_m^-$  by  $\Gamma_n^+$ .

If condition (2) of the theorem holds, it is not necessary to use (A.6). In this case the proof will be similar, being based on the following chain of inequalities (The proof is based on assuming the contrary, so that  $\Gamma_m^- \not\subseteq W \setminus \Gamma_n^+$ , i.e., there exists, as it were, an element  $\xi \in \Gamma_m^-$  and  $\xi \in \Gamma_n^+$ ):

$$\begin{aligned} \pi\Gamma_n^+(\xi) &\leq F(\Gamma_n^+) < F(\Gamma_m^-) \leq \\ &\leq \pi\Gamma_m^-(\xi) \leq \pi^-W(\xi) \leq \pi\Gamma_n^+(\xi) \end{aligned}$$

The first inequality follows from the definition of  $F(\Gamma_n^+)$ , the second follows from Condition (2) of the theorem, and the third from the definition of  $F(\Gamma_m^-)$ . The last two relations express the properties of monotonic systems. Hence in this case we have under Condition (2) also

$$\pi\Gamma_n^+(\xi) < \pi\Gamma_n^+(\xi).$$

This completes the proof of the theorem. ■ Now follows several corollaries of Theorem 2.

**Corollary 1.** If for  $n = \overline{0, q}$  the defining sequence is  $\overline{\alpha}_+$  there exists a subset  $H \subseteq W \setminus \Gamma_n^+$  such that  $F_-(H) > F(\Gamma_n^+)$ . Thus kernel  $K^\oplus$  will belong to the set  $W \setminus \Gamma_n^+$ . Indeed, since a definable set is also kernel, it follows that  $F_-(H) \leq F(\Gamma_p^-)$ ,  $m = \overline{0, 1, \dots, p}$ , and hence (in any case) if  $m = p$ , and  $n$  is selected on the basis of the condition of the corollary, then  $F(\Gamma_n^+) < F(\Gamma_p^-)$ . By virtue of the theorem, we therefore obtain the assertion of the corollary.

**Corollary 2.** If for  $n = \overline{0, 1, \dots, q - 1}$  of a defining sequence  $\overline{\alpha}_+$  there exists a subset  $H \subseteq W \setminus \Gamma_n^+$  such that  $F_-(H) = F(\Gamma_n^+)$ , then the kernel  $K^\oplus$  will belong to the set  $W \setminus \Gamma_{n+1}^+$ .

The proof follows directly from Corollary 1, by virtue of (A.6).

**Corollary 3.** If for  $m = 0, 1, \dots, p$  of a defining sequence  $\bar{\alpha}_-$  there exists a subset  $H \subseteq W \setminus \Gamma_m^-$  such that  $F_+(H) < F(\Gamma_m^-)$  then the kernel  $K^\ominus$  will belong to the set  $W \setminus \Gamma_m^-$ . The proof of Corollary 3 is entirely similar to that of Corollary 1. It is only necessary to change the signs of the inequalities and replace the set  $\Gamma_n^+$  by  $\Gamma_m^-$ .

**Corollary 4.** If for  $m = 0, 1, \dots, p-1$  of a defining sequence  $\bar{\alpha}_-$  there exists a subset  $H \subseteq W \setminus \Gamma_m^-$  such that  $F_+(H) = F(\Gamma_m^-)$ , then the kernel  $K^\ominus$  will belong to the set  $W \setminus \Gamma_{m+1}^-$ .

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