The Sugar-Pie Game: The Case of Non-Conforming Expectations *

Abstract. Playing a bargaining game, the players with non-conforming expectations were trying to enlarge their share of a sugar-pie. The first player, who was not very keen on sweets, placed an emphasis on quality. In contrast, for the second player, all sweet options, whatever they might be, were open. Thus, this paper aims to determine the negotiating power of the first player, if equal division of the pie was desirable, i.e., both players aimed to get ½ of the available sweets.

Keywords: game theory; bargaining power; non-conforming expectations

1. Introduction

When bargaining, the players are usually trying to split an economic surplus in a rational and efficient manner. In the context of this paper, the main problem the players are trying to solve during negotiations is the slicing of the pie. Slicing depends upon characteristics and expectations of the bargainers. For example, while moving along the line at the z-axis (the size), the u-axis in Fig. 1 displays single-peaked expectations of player No. 1. In comparison, concave expectations of player No. 2 are shown in Fig. 2. The elevated single-peaked $^5/_6$ -slice curve in Fig. 1 corresponds to the lower, but adversely increasing, concave $^1/_6$ curve of expectations in Fig. 2, and for the other sugar-pie allotment $^1/_9$, $^8/_9$.

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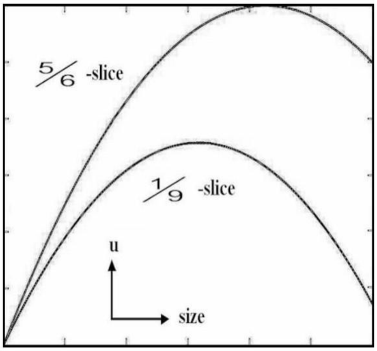

Figure 1. Player No. 1 expectations

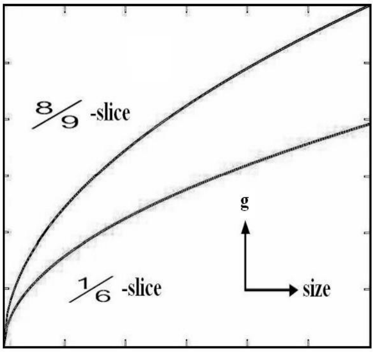

Figure 2. Player No. 2 expectations

Given that the players' expectations are non-conforming,[1] as shown in Fig 1., and Fig. 2, splitting a pie no longer represents any traditional bargaining procedure. Instead of dividing the slices, the procedures can be resettled. Thus, we can proceed at distinct levels of one parameter — parametrical interval of the size, which turns to be the scope of negotiations. In fact, Cardona and Ponsattí (2007, p. 628) noticed that "*the bargaining problem is not radically different from negotiations to split a private surplus,*" when all the parties in the bargaining process have the same, conforming expectations. This is even true when the expectations of the second player are principally non-conforming, i.e., concave, rather than single-peaked. Indeed, in the case of non-conforming expectations, the scope of negotiations — also known as "*well defined bargaining problem*" or "*bargaining set*" related to individual rationality (Roth, 1977) — allows for dropping the axiom of "Pareto efficiency." Thus, combined with the *breakdown* point, the well-defined problem, instead of slices, can be solved inside parametrical interval of the pie size.

With these remarks in mind, any procedure of negotiating on slices accompanied by sizes can be perceived as two sides of the same bargain portfolio. Therefore, it is irrelevant whether the players are bargaining on slices of the pie, or trying to agree on their size. This highlights the main advantage of the parametric procedure — it brings about a number of different patterns of interpretations of outcomes in the game. For example, it can link an outcome of an economy to a suitable size of production, scarcity of resources, etc. — all of which are indicators of most desirable solutions. Indeed, our initiative could serve to unify the theoretical structure of economic analysis of productivity problem. Leibenstein (1979, p. 493) emphasized that *"...the situation need not be a zero sum game. Tactics, that determine the division can affect the size of the pie.*" Clarifying these guidelines, Altman (2006, p. 149) wrote:

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[1] We say also interpersonally incompatible, i.e., impossible to match through a monotone transformation (Narens & Luce, 1983).

"There are two components to the productivity problem: one relates to the determination of the size of the pie, while the second relates to the division of the pie. Looked upon independently, all agents can jointly gain by increasing the pie size, but optimal pie size is determined by the division of pie size."

2. THE GAME

The game demonstrates how a sugar-pie is fairly sliced between two players. The first player, denoted as HE, is a soft negotiator, not very keen on sweets, and would not accept a pie if the size of the pie is too small or too large. In HIS view, too small or too large sugar-pies are not of reasonable quality. The second player, hereafter referred to as SHE, is a tough negotiator and prefers obtaining sweets, whatever they are.[2]

The axiomatic bargaining theory finds the asymmetric Nash solution by maximizing the product of players' expectations above the disagreement point $d = \langle d_1, d_2 \rangle$:

$$\arg\max_{0 \le x+y \le 1} f(x, y, \alpha) = (u(x) - d_1)^{\alpha} \cdot (g(y) - d_2)^{1-\alpha},$$

the asymmetric variant (Kalai, 1977).

Although the answer may be known to the game theory purists, the questions often asked by many include: *What are* x, y, α, $u(x)$ *and* $g(y)$*? What does the point* $\langle d_1, d_2 \rangle$ *mean? How is the* $\arg\max$ *formula used?* The simple answer can be given as:

- x *is HIS slicing of the pie, and* α *is HIS bargaining power,* $0 \le x \le 1$, $0 \le \alpha \le 1$;
- $u(x)$ *is HIS expectation, for example* $u(x) \equiv x$, *of HIS* x *slicing of the pie;*
- y *is HER slicing of the pie, and* $1 - \alpha$ *is HER bargaining power,* $0 \le y \le 1$;
- $g(y)$ *is HER expectation, for example* $g(y) \equiv \sqrt{y}$, *of HER* y *slicing of the pie.*

Based on the widely accepted nomenclature, we call $s = \langle u(x), g(y) \rangle$ the utility pair. The disagreement point $d = \langle d_1, d_2 \rangle$ denotes what HE and SHE collect if they disagree on how to slice the pie. The sugar-pie disagree-

[2] Note that, for the purpose of the game, we do not ignore the size of the pie but put this issue temporarily aside.

ment point is $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$, whereby the players collect nothing. Further, we believe that expectations from the pie are more valuable for HER, indicating HER desire $g(½) = \sqrt{½} = 0.707$ for sweets, which is greater than HIS desire $u(½) = 0.5$.

Now, considering the $\arg\max$ formula $f(x, y, \alpha)$, one may ask a new question: *What is the standard that will help to redesign bargaining power* α *facilitating* HIS *negotiations to obtain a desired half of the pie?* SHE may only accept or reject the proposal. A technical person can shed light on the solution. We can start by replacing $u(x)$ with x, $y = 1 - x$, $g(y)$ with $\sqrt{1-x}$, and taking the derivative of the result $f(x, 1-x, \alpha)$ with respect to the variable x by evaluating $f'_x(x, 1-x, \alpha)$. Finally, with $x = ½$, the equation $f'_x(½, ½, \alpha) = 0$ can be solved for α; indeed, $\alpha = 1/3$ provides a solution to the equation $f'_x(½, ½, \alpha) = 0$.

In general, one might feel comfort in the following judgment:

> "*Even in the face of the fact that SHE is twice as tough a negotiator,*[3] *to count on the half of the pie is a realistic attitude toward HIS position of negotiations. Surely, rather sooner than later, since HE revealed that SHE prefers sweets whatever they are, HE would have HER agree to a concession.*"

This attitude might well be the standard of redesigning the power of HIS negotiation abilities if half of the pie is desirable as a specific outcome of negotiations.

Returning to the pie size issue, it will be assumed that, in the background of HIS judgment, the quality of the pie first increases, when the size is small. On the other hand, as the size increases, the quality eventually reaches the peak point, after which it starts to decline with the increasing size. Thus, the quality is single-peaked with respect to the size. For HER, the pie is always desirable. To handle the situation, we assume that HE possesses all the relevant skills of the pie slicing. Nonetheless, based on the aforementioned assumptions, for HIM, the slicing may, in some cases, not be worth the effort at all. If the slicing does not meet its goal, as just emphasized, HE can promote HIS own understanding of how to slice the pie properly. HE can enforce decisions, or effectively retaliate for breaches — recruiting for example "enthusiastic supporters," (Kalai, 1977, p.131). SHE, on the other hand, lacks slicing abilities, knowledge, skills or competence. Thus, if interests of both players in the final agreement are sometimes different or sometimes not, SHE cannot fully control HIS actions and intentions. In these circumstances, SHE might show a willingness to agree with HIS pie division, or at least not resist HIS privileges to make arrangements upon the size of the pie. Hence, from HER own critical point of view, by acting in common interest, SHE may admit HER lack of knowledge and skill. This clarifies HIS and HER asymmetric power dynamics.

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3 Let us say that SHE pays HER solicitor twice as much as HE does.

Whether HE is committed or not is irrelevant for his decision to accept HER recommendation regarding the size z. HE is committed, however, only to slice x aligned in eventual agreement. The above can be restated, then, with the condition that HE seeks an efficient size z of the pie determined by the slice x. Let, as an example, the utility pair $\langle u, g \rangle$ of HIS and HER expectations be given by:

$$u(z,x) = z \cdot [(1 + x/2) - z] \; ;$$
$$g(z,y) = z \cdot \sqrt{y}, \; z \in [0,1], \; x, y \in [0,1].$$

The root $z = \frac{1}{2}$ resolves $\langle u'_z(z,x)|_{x=0} \rangle = 0$ for z, and the root $z = \frac{3}{4}$ resolves $\langle u'_z(z,x)|_{x=1} \rangle$ accordingly. We can thus define efficient slices, relative to the size z, as a curve $x(z)$, which solves $u'_z(z,x) = 0$ for x. Evaluating x from $u'_z(z,x) = 0$ and subsequently replacing $x(z)$ into $u(z,x)$ and $g(z,x)$, yields $u(z) = z^2$ and $g(z) = z \cdot \sqrt{3 - 4 \cdot z}$. Now, given the scope $z \in [\frac{1}{2}, \frac{3}{4}] \subset [0,1]$ of the negotiations, the bargaining problem $\langle \boldsymbol{S}, d \rangle$ passes then into parametric space $\boldsymbol{S}_z = \langle u(z), g(z) \rangle$. In HIS view, the pie must fit the size requirements, since outside the interval $[\frac{1}{2}, \frac{3}{4}] \subset [0,1]$ the size z is inefficient — too small and thus not useful at all, or too large and of inferior quality. Therefore, the disagreement occurs at $d = \langle u(\frac{1}{2}), g(\frac{3}{4}) \rangle$, $d = \langle \frac{1}{4}, 0 \rangle$. The Nash symmetric solution to the game is found at $z = 0.69$, $x = 0.74$. On the other hand, HIS asymmetric power 0.21 is adequate for negotiating with HER about receiving half of the pie. The size $z = 0.62$, for example, in HIS view, fits the necessary capacities of a *stovetop* for provision of high quality sugar-pie.

Once again, to find the Nash symmetric solution, a technically minded person must resolve the equation $f'_z(z,\alpha) = 0$ *for* z, *where* $f(z,\alpha) = (u(z) - \frac{1}{4})^{\alpha} \cdot g(z)^{1-\alpha}$ *when* $\alpha = \frac{1}{2}$; $z = 0.69$ *provides a solution to the equation. Thus, solving the equation* $u'_z(0.69, x) = 0$ *for* x *yields* $x = 0.74$. *To find the power of asymmetric solution, we first solve the equation* $u'_z(z, \frac{1}{2}) = 0$ *for* z, $z = 0.62$, $x = \frac{1}{2}$. *Then, we solve* $f'_z(0.62, \alpha) = 0$ *for* α *and find that HIS power matches* $\alpha = 0.21$, *which is adequate for negotiating with HER when an equal slicing of the pie is desirable, i.e., both HE and SHE receive ½ of the pie.*

3. BARGAINING PROCEDURE

The strategic bargaining game operates as a game of alternating offers. Given some light conditions, it is well known that, when players partaking in this type of game are willing to make concessions during the negotiations, they are likely to embrace the axiomatic solution. That is the reason why we continue our discussion in terms of a procedure similar to the strategic approach.

To recall, there are two players in our game — HE, with emphasis on quality, and SHE, with no specific preferences. A precondition for the agreement was that the expectations of negotiators solely depend on HIS framework of how to set the size parameter, rather than the slice. As a consequence of this dependence, efficient sizes provide a fundamental correspondence to crucial slices. Accepting the precondition, SHE will only propose efficient sizes, as HE will reject all other choices.

Nonetheless, it is realistic that SHE would — by negligence, mistake or some other reason — recommend an inefficient size, which HE would mistakenly accept. On the contrary, it is also realistic that HE has an intention to disregard an efficient recommendation. This will be irrational handling as, in any agreement, no matter what is going on, both players are committed by proposals to slices. Therefore, making a new proposal, HER recommendation on sizes makes a rational argument that HE must accept or reject in a standard way. Such an account, instead of an agreement upon slices, as we believe, explains that the outcome of the bargaining game might be a desirable size $z^{o} \in [z_1, z_2]$. Hereby, only the interval, named also the scope $[z_1, z_2]$ of negotiations, bids proposals, which now, by default, are binding efficient sizes with slices x. Consequently, the bargaining game performs exclusively in the interval $[z_1, z_2]$. Hence, $[z_1, z_2]$ is the scope of HIS efficient sizes of most trusted sugar-pie platforms for negotiations, where players would choose sizes, accepting or rejecting proposals. The negotiators' expectations, depending on $[z_1, z_2]$, arrange a bargaining frontier $\boldsymbol{S}_z$ as a way to assemble the bargain portfolio. Therefore, the negotiators may focus on making the size proposals. If rejected, the roles of actors change and a new proposal is submitted. The game continues in a traditional way, i.e., by alternating offers.

Observation. *In the alternating-offers sugar-pie game, the functions* $\left(u(z)-d_1\right)^{\alpha}$ *and* $\left(g(z)-d_2\right)^{1-\alpha}$ *imply HIS and HER expectations, respectively, over the pie size* $z\in\left[z_1,z_2\right]$. *With the risk* $1>>q>0$ *of negotiations to collapse prematurely into disagreement point* $d=\left[d_1,d_2\right]$, *the solution* z^{o} *of well-defined bargaining problem* $\left\langle \boldsymbol{S}_z,d\right\rangle$ *is enclosed into the interval* $\left[z',z''\right]\subset\left[z_1,z_2\right]$, $z^{o}\in\left[z',z''\right]$. *The margins* z',z'' *are solving the equations*

$$(1-q)\cdot\left(u(z^1)-d_1\right)^{\alpha}=\left(u(z^2)-d_1\right)^{\alpha},$$

$$(1-q)\cdot\left(g(z^2)-d_2\right)^{1-\alpha}=\left(g(z^1)-d_2\right)^{1-\alpha}$$

for variables z^1,z^2 (cf. Rubinstein 1998, p. 75).

In our example, when $x=\frac{1}{2}$ *(the half of the pie is a desirable (ex-ante) solution), HIS negotiation power* 0.21 *leads to the asymmetric solution* $z=0.62$. *Let the risk factor of the premature collapse of negotiators be* $q=0.05$. *Then, the interval* $\left[0.61,0.64\right]\subset\left[0,1\right]$ *sets up pie sizes providing the desirable solution, whereby the pie will be divided equally.*

4. Conclusion

In view of the above, a pretext for the analysis of the domain and the extent of bargain portfolio for two fictitious negotiators, denoted as HE and SHE, were established. The portfolio was supposed to account for the players having non-conforming expectations. Instead of slicing the sugar-pie, such an account allowed for the inclusion of a guide on how the eventual consensus ought to be analyzed and interpreted within the scope of negotiations upon the size of the pie. Players' bargaining power indicators specified by the bargaining problem solution were used in compliance with their respective desired visions and ambitions.

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