Equilibrium in a Retail Chain with Transaction Costs: Rational Coalitions in Monotonic Games *

Abstract. In a given context, a situation is considered when a retail chain of suppliers, agents and distributors transforms while transaction costs increase. As costs increased, orders and deliveries between relevant chain's groups resulted in the most cost-resilient retail chain. The participants in such a resilient chain remain in equilibrium, provided that in any transaction, the profit from trading exceeds the cost of the transaction, including transportation costs. In making decisions about buying and selling, the participants in the chain had to follow the rules and regulations of what the author called a monotonous game. A formal scheme of coalition formation in this monotonic game of connected retail trade participants with monotonic utility functions is described. Special coalitions are studied that have an advantage for each of the participants over the rest in the sense of a greater ability to withstand the volatility of the supply market.

Keywords: suppliers; distributors; monotonic game; retail chain; coalition.

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Businessmen in deciding on their ways of doing business and on what to produce have to take into account transaction costs. If the cost of making an exchange are greater than the gains which that exchange would bring, that exchange would not take place and the greater production that would flow from specialization would not be realized. In this way transaction costs affect not only contractual arrangements, but also what goods and services are produced. Ronald H. Coase, "The Institutional Structure of Production," Ménard, C., and M. M. Shirley (eds.) (2005), *Handbook of New Institutional Economics*, Spriner: Dordrecht, Berlin, Heidelberg, New York. XIII. 884pp., p.35, ISBN 1-4020-2687-0.

1. INTRODUCTION

Everybody, probably knows that prices on commodity markets sometimes continue to rise unabated on the back of an anticipated shortage in the global raw materials availability and sharp volatility in the commodity future markets and terminal prices on fears of an immediate shortage of materials in the short term. Along with the significant increase in commodity prices, on one hand, the transaction costs increase on inputs like petroleum, electricity, etc. On the other, while currency of exchange rates also moving adversely, the situation becomes uncertain. As an example, one may point at recent market price increase of coffee raw materials, which did not have immediate consequences for some known positions, while the distributors [1] of a retail chain, however, demonstrate readiness to make loosing transactions. With this in mind, distributors are trying to hold prices constant. However, it is also understandable that it would be impossible for the distributor to make frequent price changes again and again. Given the current context, they will have no other option but to seek price increase for distributed commodities with an immediate effect.

Uncertainties in market prices of commodities always lead to an increase of transaction costs. Transaction costs increase once again leads to additional uncertainties, and the distributors in the retail chain end up in a dead circle of price increase, which may result that the bilateral trade does not take place, and the market old supply and demand structure to be replaced with a new. In the environment of constant price increase, the orders and deliveries do not match any more for a given supply and demand structure. In such situations, individual participants in the retail chain are still assumed to act rationally finding a new ways of making business with the object of maximizing the profit by trying to restructure the chain. Worth to note that New Institutional Economics gives an explanation for transactions as mediated through the market in two directions: the vertical integration, Joskow (2005), where the market structure is mostly a vertical chain of semi-product components, and the horizontal chain of services and products outsourced by companies if needed to produce the end product.

This paper addresses the above situation in question by setting up a retail chain game of the participants in the chain grounding on supposition that orders and deliveries be met with uncertainty of transaction costs. In so doing, the

[1] A group of retail outlets owned by one firm and spread nationwide or worldwide.

paper attempts to develop a numerical description of the supply and demand structure for the deliveries of commodities in the retail chain. The allegedly rational behavior of a participant is not always such, because the participants on purpose may attempt to enter but irrationally into certain losing transactions in hope to offset the negative effect of the former. Given this irrational situation the prices will increase additionally upon already profitable transactions. Numerical analysis of irrational situations reveals, however, that in case the participants will try to avoid all losing transactions, their behavior is once again becoming rational and in such situations the participants of the retail chain will end up in the Nash equilibrium (1953).

To our knowledge (or lack of that), the retail chain formation, or in mundane terms the restructuring process of the retail chain is rather complicated mathematical problem, which do not have satisfactory solutions. However, in recent years it has become clear that a mathematical structure known as antimatroid is well suited for such type a retail chain formation process (cf. Algaba, et al, 2004). Antimatroid is a collection of potential interests groups — subsets of participants, i.e., those who make decisions to buy and sale in bilateral trade transactions. That is to say, within antimatroid one will always find a path of transactions connecting members of the retail chain — if the latter forms of course — with each other by mutual business interests inside groups/coalitions belonging to antimatroid and making the exchange as participants of a characteristic retail chain.

We step up beyond convention of the theory of coalition games that the solution mandatory has to be a core, and take the retail chain formation process in terms of so-called *defining sequence* of transactions (Mullat, 1979). The sequence facilitates the retail chain formation as a transformation process of nested sets of bilateral transactions, which ends at its last and highest costs' threshold — the most tolerant retail chain towards costs — a kernel. Hereby, the kernel operates as a retail chain of participants capable to cover the highest transaction costs in case of uncertainty. In our case, the *defining sequence* of transactions produces the elements of an antimatroid — some interest groups, cf. Levit and Kempner, (2001); see also (1991) Korte et al. The defining sequence on antimatroid, in particular, follows the Greedy heuristic procedure of Shapley's value, but in inverse order, cf. Rapoport (1985).

Bearing all this in mind, the suggested framework allows performing a series of computer simulations. First, to determine the possible response of the retail chain participants, to different supply and demand structures. Second, to identify the participants, where the executive efforts might be applied to prevent unpredictable actions that may misbalance the equilibrium in the retail chain. With this object, we used a model to assemble an "elasticity" measure for the choice of customers; this measure is represented by transaction costs' interval, for which the retail chain remains in equilibrium.

The rest of this paper is structured as follows. The next section sets up the basic concepts intending to bring at the surface the calculus of utilities of participants in the retail chain. It is a preliminary step necessary to move forward to the Section 3, where the general model of participants of the chain is described. In Section 4, which is main part of the paper, the retail chain game of customers addresses the process of the chain formation in details. Here the monotonic property of utilities plays its major role. In Sections 5-6, we construct different varieties of coalitions of retail network players that are "outstanding" in the sense of rationality, and indicate relations between such coalitions. Also, constructive processes described in Section 7 for discovering these *outstanding players*, described in additional Section 8. A summary of the results ends the study. The proofs of all theorems, etc., ... are given in the Appendix.

2. DESCRIPTION OF A RETAIL CHAIN: THE SIMPLE FORM

To consider the simplest case of commodities distribution in a retail chain might be instructive. This elementary model is used at current stage solely as a convenient means of simplifying the presentation.

The distribution of commodities in the retail chain is characterized by sales figures that may be expressed as one of the following three alternative numbers: a) a demand η which is disclosed to the particular participant either externally or by other participant in the chain; b) a capable supply ξ calculated at the cost of all commodities produced by the participant for delivery outside the chain or to the other participants; c) actual sales γ calculated at the prices actually paid by the customers for the delivered commodities.

An order is thus defined as a certain quantity of a particular commodity ordered by one of the participant's from another participant in the retail chain; a delivery is similarly defined as a certain quantity of a commodity delivered by one of the participant's to another participant in the chain. We assume that the chain includes suppliers who are only capable of making deliveries – the produces; participants, who both issue orders and make deliveries – the agents; and the distributors, who only order commodities from other participants. [2]

In what follows we consider the retail chain of orders and deliveries for the case like "pipeline" distribution without "closed circuits." Therefore, we can always identify a unique direction of "retail chain" of orders from the distributors to the produces via agents and a "retail chain" of deliveries in the reverse direction.

Let us consider in more detail this particular retail chain of orders and deliveries of commodities. The direction of the chain of orders (deliveries) is defined by assigning serial numbers – the indexes $1, 2$ and 3 – to the producer, to the agent, and to the distributor, respectively. The producer and the agent act as suppliers, the agent and the distributor act as customers. The agent thus has the dual role of a supplier and a customer, whereas the producer only acts as a supplier and the distributor only acts as a customer.

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[2] The distributors also act as suppliers to external customers.

The chain of orders to the produces from the customers is characterized by two numbers η_{23} and η_{12}. The number η_{wj} $(w = 1,2; j = 2,3)$ is the demand η_{wj} disclosed by the customer j to the supplier w. We assume that sales are equal to deliveries. Two numbers ξ_{12} and ξ_{23}, which are interpreted as the corresponding capable sales similarly characterize the chain of deliveries to the distributor.

Suppose that the demand of the distributor to the external customers is fixed by d bank notes. The capable sales of the producer are s bank notes. In other words, d is the estimated amount of orders from the external customers and it plays the same role as the number η for the customers in the retail chain. Similarly, s is the intrastate amount of estimated deliveries by the producer, and it has the same role as ξ for the customers.

Let us now consider the exact situation in a chain. To make deliveries at a demand amount of d bank notes, the distributor have to place orders with the agent in the amount of $\eta_{23} = \nu_{23} \cdot d$ bank notes, where ν_{23} are the distributor's cost of commodities sold (the cost per 1 bank note of sales). The agent, having received an order from the distributor, will in turn place an order with the supplier in the amount $\nu_{12} \cdot \eta_{23}$, where ν_{12} is the agent's cost per one bank note of sales. On the other hand, the estimated sales of the producer are ξ_{12} bank notes, $\xi_{12} = s$. Assuming that all the transactions between the suppliers and the customers in the retail chain are materialized in amounts not less than those indicated in the purchase orders, the actual sales of the producer to the agent are given by $\gamma'_{12} = \min\{\xi_{12}, \eta_{12}\}$.

Now, since the agent paid the producer γ'_{12} for the commodities ordered, the agent's revenue is $\xi_{23} = \gamma'_{12}/\nu_{12}$, where clearly $\xi_{23} \geq \gamma'_{12}$. The difference between the revenue ξ_{23} and the costs γ'_{12} is defined as

$$\pi_{12} = \gamma'_{12} \cdot (1 - \nu_{12})/\nu_{12} .$$

From the same considerations, $\gamma'_{23} = \min\{\xi_{23}, \eta_{23}\}$ [3]give the actual sales of the agent to the distributor. We similarly define the difference $\pi_{23} = \gamma'_{23} \cdot (1 - \nu_{23})/\nu_{23}$. The numbers π_{12}, π_{23} represent the profit of the customers in the retail chain.

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[3] In subsequent sections, γ'_{wj} is replaced by $\gamma_{wj} = \gamma'_{wj}/\nu_{wj}$. The numbers γ and γ' differ in the units of measurement of the commodities delivered to the user j. While γ' represents the sales at the cost, γ represents the same sales at actual selling prices.

In conclusion of this section, let us consider the numbers π_{12}, π_{23} more closely. We see from the above discussion that the material costs are the only component of the costs of commodities sold for the customers in the retail chain; no other producing or transaction costs are considered. And yet in Section 4 the numbers π_{12}, π_{23} are used as the admissible bounds on transaction costs, which are assumed to be unknown. It is in this sense we construct a model of a monotonic game of customers (Mullat, 1979, p.6).

3. Description of a Retail Chain: the General Form

Consider now a retail chain consisting of n participants indexed w, $j = 1,2,...,n$. The state of a supplier w is characterized by a $(m+1)$-component vector [4] $\langle d_w, y_w \rangle = \langle d_w, \eta_{wk+1,...,}\eta_{wn} \rangle$, $(n-k=m)$; the state of a customer j by a $(v+1)$-component vector $\langle s_j, x_j \rangle = \langle s_j, \gamma_{1j},...,\gamma_{vj} \rangle$. The components of the $\langle d_w, y_w \rangle$ and $\langle s_j, x_j \rangle$ vectors are interpreted as follows: d_w is the total orders amount of the supplier w acting as a customer; s_j is the capable sales total amount of the customer j acting as a supplier; η_{wj} is the cost of orders placed by the customer j with the supplier w; γ_{wj} are actual sales (deliveries) to customer j from the supplier w. As indicated in the footnote, γ_{wj} represents the deliveries valued at the selling prices of the customer j acting as a supplier. The vectors $\langle d_w, y_w \rangle$, $\langle s_j, x_j \rangle$ are the order and the delivery vectors, respectively.

With each participant in the retail chain we associate certain domains in the nonnegative orthants $\mathfrak{R}^{m+1}$ of the $(m+1)$ – and $\mathfrak{R}^{v+1}$ of the $(v+1)$ – dimensional space. These domains $\mathfrak{R}^{m+1}$ and $\mathfrak{R}^{v+1}$ are the regions of feasible values of vectors $\langle d_w, y_w \rangle$, $\langle s_j, x_j \rangle$ in the $(m+v+2)$ – dimensional space.

For some of the participants vectors with $\gamma_{wj} > 0$ are inadmissible, and for some participants vectors with $\eta_{wj} > 0$ are inadmissible. Participants having the former property will be called produces and those having the latter property will be called distributors; all other participants in the retail chain will be called

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[4] k is the number of produces, see below.

agents. In what follows the numbers s_w ($w = 1,2,...,k$) characterize the k produces; the number s_w represents the capable sales controlled by the participant w. The numbers d_j ($j = v+1, v+2,...,n$) correspondingly characterize the r distributors: the number d_j represents the demand to the external customers ($n - v = r$).

Let us now impose certain constrains on the admissible vectors in this retail chain. The following constrains are strictly "local," i.e., they apply to the individual participants in the retail chain.

The admissible retail chain states are constrained by balance conditions equating the actual sales from all the suppliers to a particular customer to capable sales of that customer acting as a supplier:

$$s_j = \sum_{w=1}^{v} \gamma_{wj} \quad (j = k+1, k+2,...,n). \tag{1}$$

We also require balance conditions between the cost of orders placed by all the customers with a particular supplier and the demand figure of that supplier acing as a customer:

$$d_w = \sum_{j=i+1}^{n} \eta_{wj} \quad (w = 1,2,...,v). \tag{2}$$

As we have noted above, the retail chain considered in this article does not allow "closed-circuit motion" of orders or deliveries until a particular order reaches a producer or the delivery reaches a distributor. The indexes labeling the participants in such chains are ordered in a way [5] that if w is a supplier and j is a customer, then $w < j$ ($w = 1,2,...,v;\ j = v+1, v+2,...,n$). We call such chains as of a retail-type, and their description requires certain additional assumptions.

Consider the constants $\alpha_{wj} \geq 0$ and $\beta_{wj} \geq 0$ satisfying the following constraints ($w < j; j = k+1,...,n$):

$$\sum_j \alpha_{wj} \leq 1 \ (j > w; w = 1,2,...,v), \ \sum_w \beta_{wj} \leq 1 \tag{3}$$

For the supplier w, the number α_{wj} is the fractional cost of orders made

[5] The term topological sorting originates from Knuth (1972) to describe the ordering of indexes having this property.

to the customer j. For customer j, the number $\eta_{wj} = \beta_{wj} \cdot d_j \cdot \nu_{wj}$ is the fractional cost of the deliveries from supplier w, which are necessary for meeting the sales target.

Suppose that purchase of orders in the retail chain move from distributors through agents to suppliers. This chain is conducted at the wholesale prices. The deliveries, also conducted at the wholesale prices of the chain in the opposite direction. We express the effective wholesale prices by a set of constants ν_{wj} ($w = 1,2,...,v; j = k+1, k+2,...,n$), which represent the participant's cost per one bank note of sales for a customer acting as a supplier.

The set of constants α_{wj}, β_{wj} and ν_{wj} make it possible to uniquely determine the amount of orders and deliveries in a given transaction. Indeed, the amount of orders to the supplier w from the customer j is given by $\eta_{wj} = \beta_{wj} \cdot d_j \cdot \nu_{wj}$. The relation (see Section 2) determines the amount of deliveries $\gamma'_{wj} = \min\{\xi_{wj}, \eta_{wj}\}$, where $\xi_{wj} = s_w \cdot \alpha_{wj}$ are the capable sales values at cost prices. Considering the difference in revenue from sales of customer j acting as a supplier, we conclude that the deliveries from the supplier w to the customer j are given by $\gamma_{wj} = \gamma'_{wj} / \nu_{wj}$.

In conclusion, let us consider one computational aspect of order and delivery vectors in a retail-type distribution chain.[6] It is easily seen that the components d_j, s_w, η_{wj} and γ_{wj} $(w = 1,2,...,v;\ j = k+1, k+2,...,n)$ as obtained from (1) and (2) are given by ($w < j; j = k+1,...,n$)

$$d_w = \sum_j \beta_{wj} \cdot d_j \cdot \nu_{wj} \quad (j > w; w = 1,2,...,v) \tag{4}$$

$$s_j = \sum_\omega \min\{s_w \cdot \alpha_{wj}; \beta_{wj} \cdot d_j \cdot \nu_{wj}\} / \nu_{wj} \tag{5}$$

The starting data in (4) is the demand of the distributors to external customers, i.e., the numbers $d_{v+1}, d_{v+2},..., d_n$. The starting data in (5) are the capable sales amounts $s_1, s_2,..., s_k$ of the produces, which together with the numbers $d_1, d_2,..., d_v$ from (4) are used in (5) to compute the actual sales of the customers.

[6] Here we need only consider the principles of the computational procedure.

4. A MONOTONIC GAME OF CUSTOMERS IN THE RETAIL CHAIN

In the previous section we considered a retail-type distribution in the chain with participants indexed by $w = 1,2,...,v;\ \ j = k+1, k+2,...,n$: the index j identifiers a customer, the index w identifiers a supplier.

Let us interpret the activity of the retail chain as a monotonic game (Mullat, 1979), in which the customers need to decide from what supplier to order a particular commodity.

Suppose that in addition to the cost of materials, the customers bear uncertain transaction costs in their bilateral trade with suppliers. Because of the uncertainty of transaction costs, it is quite possible that in some transactions the costs will exceed the gross profit from sales. In this case, the potentially feasible transactions will not take place.

Let the set R_j represents all the potential transactions corresponding to the set of suppliers from which the customer j is to make his choice. The choice of the customer j ($j = k+1, k+2,...,n$) is a subset A^j of the set R_j: $A^j \subseteq R_j$; the case $A^q = \varnothing$ is not excluded: it requires the customer's refusal to make a choice. The collection $\langle A^{k+1}, A^{k+2},..., A^n \rangle$ represents the customer's joint choice. It is readily seen that the sets R_j are finite and nonintersecting; their union corresponds to set $W = R_{k+1} \bigcup R_{k+1} \bigcup ... \bigcup R_n$.

In what follows, we focus on the criterion by which the customer j chooses his suppliers A^j while the lowest transaction costs, as a threshold u^o, increases. In contrast to the standard monotonic game (Mullat, 1979), which is based on a coalition formation, we will consider the strategy of individual customers whose objective is to maximize the profit from the actual sales revenues. We will thus essentially deal with m players' game, $m = n - k$.

Let us first introduce a measure of the utility of a transaction between customer j and supplier $w \in A^j$ ($j = k+1, k+2,...,n$). The utility of a transaction between customer j and supplier w is expressed by the corresponding profit $\pi_{wj} = \gamma_{wj} \cdot (1 - \nu_{wj})$.

The utility of a transaction with a supplier $w \in A^j$ is a function $\pi_{wj}(X_{k+1}, X_{k+2},..., X_n)$ of many variables: the value of the variable X_j is the choice A^j of the customer j, the number of variables is $m = n$ –k. To establish this fact, it is sufficient to show how to compute the components of the order and delivery vectors from the joint choice $\langle X_{k+1}, X_{k+2},..., X_n \rangle$. Indeed, according to our description, a retail-type distribution in the chain requires defining the constants $\alpha_{wj} \geq 0$ and $\beta_{wj} \geq 0$ ($w = 1,2,..., v; j = k+1,..., n$) that satisfy the constraints (3). A pair of constants α_{wj} and β_{wj} can be assigned in a one-to-one correspondence to a supplier $w \in R_j$, rewriting (3) in the form

$$\sum_{w \in R_j} \alpha_{wj} \leq 1, (w = 1,2,..., v), \sum_{w \in R_j} \beta_{wj} \leq 1, (j = k+1,..., n) \qquad (6)$$

If the constrains (6) are satisfied, then the same constrains are of necessity satisfied on the subsets A^j of the set R_j. Thus, restricting (4) and (5) to the sets $X_j \subseteq R_j$, the numbers γ_{wj} can be uniquely calculated for every joint choice $\langle X_{k+1}, X_{k+2},..., X_n \rangle$. Finally, let us define the individual utility criterion of the customer j in the form:

$$\Pi_j = \sum_{w \in A^j} (\pi_{wj} - u_{wj}), \qquad (7)$$

where u_{wj} are the customer j transaction costs allocable to the supplier $w \in A^j$; we define $\Pi_j = 0$ if the customer j refused to make a choice — $A^j = \varnothing$. The function $\pi_{wj}(X_{k+1}, X_{k+2},..., X_n)$ has the obvious property of monotone utility, so that for every pair of joint choices of customers $\langle L^{k+1}, L^{k+2},..., L^n \rangle$ and $\langle G^{k+1}, G^{k+2},..., G^n \rangle$ such that $L^j \subseteq G^j$ ($j = k+1,..., n$) we have

$$\pi_{wj}(L^{k+1}, L^{k+2},..., L^n) \leq \pi_{wj}(G^{k+1}, G^{k+2},..., G^n). \qquad (8)$$

The property of monotone utility leads to certain conclusions concerning the behavior of customers depending on the individual utility criterion. Under certain conditions, rational behavior of customer j (i.e., maximization of the profit Π_j) is equivalent to avoid profit-loosing transaction with all the suppliers $w \in A^j$. This aspect is not made explicit in Mullat (1979), although it is quite obvious. Thus, using the lemma, see the English version at p.1473, we can easily show that if the utilities $\pi_{wj}(...,X_j,...)$ are independent of the choice X_j, the customer j maximizes his profit Π_j by extending his choice to the set-theoretically largest choice. In what follows we will show that this result also applies under a weaker assumptions.

Below we first start with a few reservations about the proposed condition – see (9). This condition has a simple economic meaning: the customer j entering into loosing transactions cannot achieve a net increase in his utility of the losses. For example, if for fixed choices of all other customers in the retail chain, the utilities $\pi_{wj}(...,X_j,...)$ for $w \in X_j$ are independent of the choice X_j, the condition (9) hold as strict inequalities. These conditions are also reduces to strict inequalities when, for instance, the capable sales ξ_{wj} in each transaction between customer j and supplier $w \in A^j$ is not less than the demand η_{wj} so that every customer can receive the entire quantity ordered from his suppliers. In particular, by increasing the producers' supply $s_1, s_2, ..., s_k$ with unlimited manufacturing capacity, we can always increase the capable sales to such an extent that it exceeds the demand, so that the conditions (9) are satisfied.

We can now formulate the final conclusion: the following lemma suggests that each customer will make his choice so as to maximize the profit Π_j, providing all the other customers keep their choices fixed. [7]

Let the suppliers not entering the set A_j be assigned indexes $q = 1,2,...$. Then the profit Π_j of customer j is represented by a many-variable function $\Pi_j(t_{1j}, t_{2j},...)$ with variables t_{qj} varying on $[0, \beta_{qj}]$. [8] The value of the function $\Pi_j(t_{1j}, t_{2j},...)$ is the customer's profit for the case when the customer j has extended the choice by placing orders in the amounts of $t_{qj} \cdot d_j \cdot \nu_{qj}$ with the suppliers $q = 1,2,...$ outside the choice A_j. Thus, the customers j who expand their choice A_j, identify the suppliers

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7 The joint choice of users having this property is generally interpreted in the sense of Nash equilibrium (1953); see also Owen (1968).

8 We recall that β_{qj} is the fractional cost of all the orders placed with supplier q.

$q = 1,2,...$ by the set of variables t_{qj}. If all $t_{qj} = 0$, the choice A_j is not expanded and the profit $\Pi_j(0,0,...)$ coincides with (7).

The profit function $\Pi_j(t_{1j}, t_{2j},...)$ thus has to satisfy the following constraint: for every t_{qj} in $[0, \beta_{qj}]$ $q = 1,2,...$

$$\Pi_j(t_{1j}, t_{2j},...) \le \Pi_j(0,0,...). \quad (9)$$

Definition. *A joint choice* $\left\langle A_o^{k+1},..., A_o^n \right\rangle$ *of the retail chain customers is said to be rational with the threshold* u^o *if, given an amount of transaction costs not less than* $u^o > 0$ *, the utility measure* $\pi_{wj} \ge u^o$ *in every transaction of customer* j *with the supplier* $w \in A_o^j$*,* $j = k+1,...,n$.

Lemma. *The set-theoretically largest choice* $S^o = \left\langle A_o^{k+1},..., A_o^n \right\rangle$ *among all the joint choices rational with threshold* $u^o > 0$ *ensures that the retail-type distribution chain is in equilibrium relative to the individual profit criterion* Π_j *under the following conditions: a) the transaction costs* u_{wj} *for* $w \in S^o$ *do not exceed min* π_{wj} *over* $w \in S^o \bigcap R_j$*; b) inequality (9) holds.*

Proof. Let S^o be a set-theoretically largest choice among all the joint choices rational with the threshold u^o, i.e., S^o is the largest choice *H* among all the choices such that $\pi_{wj}(H \bigcap R_{k+1},..., H \bigcap R_n) \ge u^o$. Suppose that some customer *p* achieves a profit higher than Π_p by making the choice $A^p \subseteq R_p$, which is different from $S^o \bigcap R_p$; $\Pi_p' = \sum_{w \in A^p} \left(\pi_{wp}(..., A^p, ...) - u_{wp}\right) > \Pi_p$, subject to $u^o \le u_{wp} \le \min_{w \in A^p} \pi_{wp}$. Clearly, the choice A^p is not a subset of S^o, since this would contradict the monotone property (8), so that $A^p \setminus S^o \ne \emptyset$. By the same monotone property, the customer making the choice $A^p \bigcup \left(S^o \bigcap R_p\right)$ will achieve a profit not less than Π_p'. On the other hand, all transactions in $A^p \setminus S^o$ are losing transactions for this customer, since S^o is the set-theoretically largest set of non-losing bilateral trade agreements tolerant towards the transactions costs' threshold $u^o > 0$. For the customer p making the choice $A^p \bigcup \left(S^o \bigcap R_p\right)$ the profit Π_p' does not decrease only if the total increase in utility due to the contribution π_{wp} of the transactions

$w \in S^o \bigcap R_p$ exceeds the total negative utility due to the transactions in $A^p \setminus S^o$. Clearly, because of the constraint (9), the customer p has no such an opportunity. This contradiction establishes the truth of the lemma. ■

In conclusion, we would like to consider yet another point. With uncertain transaction costs, the refusal to enter into any transaction may lead to an undesirable "snowballing" of refusals by customers to choose their suppliers. It therefore seems that customers will attempt at least to conclude transactions with $\pi_{wj} \geq u^o$; even when there is some risk that the transaction costs will exceed the utility π_{wj}. Thus, without exaggeration, we may apparently state that the size of the interval $[u^o, \min \pi_{wj}]$ reflects the elasticity of the customer's choice: the number $\min \pi_{wj} - u^o$ is thus a measure of a "risk" that the customer will get into non-equilibrium situation. Clearly, a customer with a small interval will have greater difficulties to maintain the equilibrium than a customer with a wide interval.

5. Rational Coalitions in Monotonic Games

In many-persons games (Owen, 1971) by a coalition we shall understand a subset of participants. Among all coalitions we usually single out rational coalitions — a participant in such coalition extracts from the interaction in the coalition a benefit, which satisfies him. In addition, sometimes it is further stipulated that extraction of this benefit is ensured independently of the actions of the players not entering into the coalition.

The class of games proposed in this paper is subjected to an additional monotonic condition, which has been studied earlier in Mullat (1976, 1977) (although knowledge of the latter is not presupposed). There is no difference between the formal scheme of the present paper and that of Mullat in essence; the difference involved in interpretation is in abstract indices of interconnection of elements of the system, which are understood as utility indices. The approach developed enables us to establish, in one particular case, the possibility of finding rational coalitions in the state of individual equilibrium according to Nash.

6. FORMAL DEFINITIONS AND CONCEPTS

We consider a set of n players denoted by I. Each player $j \in I$ $(j = \overline{1,n})$ is matched by a set R_j from which the player j can select elements. It is assumed that the sets R_j are finite and do not intersect. Their union forms a set $W = R_1 \cup R_2 \cup \ldots \cup R_n$. The elements selected by the player j from R_j compose a set $A^j \subseteq R_j$. The set A^j is called the choice of the player j, while the collection $\left\langle A^1, A^2, \ldots, A^n \right\rangle$ is called the joint choice. The case $A^k = \varnothing$ is not excluded and is called the refusal of k-th player from the choice.

We introduce the utility functions of elements $w \in A^j$. We assume that certain joint choice $\left\langle A^1, A^2, \ldots, A^n \right\rangle$ has been carried out. Let there be uniquely determined, with the respect to the result of the choice, a collection of numbers $\pi_w \geq 0$ that are assigned to the elements $w \in A^j, j = 1,2,\ldots,n$; on the remaining elements of W the numbers are not determined. The numbers π_w are called utility indices, or simply utilities, and by definition, are in general case functions $\pi_w(X_1, X_2, \ldots, X_n)$ of n variables. The value of the variable X_j is the choice A^j of the player j. We shall single out utility functions possessing a special monotonic property.

Definition 1. *A set of utilities* π_w *is called monotonic, if for any pair of joint choices* $\left\langle L^1, L^2, \ldots, L^n \right\rangle$ *and* $\left\langle G^1, G^2, \ldots, G^n \right\rangle$ *such that* $L^j \subseteq G^j$,

$$\pi_w(L^1, L^2, \ldots, L^n) \leq \pi_w(G^1, G^2, \ldots, G^n) \qquad (10)$$

is fulfilled for any $w \in L^j$[9]. $j = 1,2,\ldots,n$

We now turn to the problem of coalition formation. We shall call any nonempty subset of the set of players a coalition. Let there be given a coalition V, and let its participants have made their choices. We compose from the choices A^j of the participants of the coalition V a set-theoretic union H, which is called the choice of the coalition V: $H = \bigcup_{j \in V} A^j$.[10]

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[9] We note that fulfilment of (1) is not required for the element $w \notin L^j$. Furthermore, even the numbers π_w themselves may not be defined for $w \notin L^j$.

[10] A choice H without indication about the coalition V, which has affected it, is not considered, and if somewhere the symbol V is omitted, then under a coalition we understand a collection of players such and only such for which $H \cap R_j \neq \varnothing$.

To determine the degree of suitability of the selection of an element $w \in R_j$ for the player j, a participant of the coalition, we introduce an index of guaranteed utility. With this aim we turn our attention to the dependence of the utility indices on the choice of the players not entering into coalition. It is not difficult to note that as a consequence of the monotonic condition of the functions π_w the worst case for the participants of the coalition will be when all players outside the coalition V reject the choice: $A^k = \varnothing$, $k \notin V$, so that all elements outside H will not be chosen by any of the players who are capable of making their choices. In other words, the guaranteed (the least value) of utility π_w of an element *w* chosen by a player in the case of fixed choices $H \cap R_j$ of his partners in the coalition equals $\pi_w(H \cap R_1, ..., A^j, ..., H \cap R_n)$.

The quantity

$$g_j(H) = \min_{w \in A^j} \pi_w(H \cap R_1, ..., A^j, ..., H \cap R_n)$$

is called the guarantee of the participant j in the coalition V for the choice H.

We assume that according to the rules of the game, for each chosen element $w \in A^j$ a player $j \in V$ must make a payment u°. [i] It is obvious that under condition of the payment u° the selection of each element $w \in A^j$ is profitable or at least without loss to the player $j \in V$ if and only if $\pi_w \geq u^\circ$. In the calculation for the worst case this thus reduces to the criterion $g_j(H) \geq u^\circ$. In reality we shall be interested, in relation to the player $j \in V$, in all three possibilities: a) $g_j(H) > u^\circ$, b) $g_j(H) = u^\circ$ and c) $g_j(H) < u^\circ$. We shall say that a participant of the coalition V is above u°, on the level of u°, and below u°, if the conditions a), b), and c) are fulfilled respectively. The size of the payment is further considered as a parameter u of the game being described and is called the threshold. We shall say that a coalition V, having made a choice H, functions on the level $u[H] = \min_{j \in V} g_j(H)$.

Definition 2. *A coalition* V *is called rational with the respect to a threshold* $u^\circ = u[H]$ *if for a certain choice* H *all participants of the coalition are not below* u° *while someone in the coalition* $k \cup V$ *is below* u° *if any participant* $k \notin V$ *outside the coalition* V *makes a nonempty choice* $A^k \neq \varnothing$.

The set of numerical values being attained by the function $u[H]$ on rational coalitions will be called the spectrum. Each value of the function $u[H]$ will be called the spectral level (or simply the level). The entire construction described above will be called a monotonic parametric game on W.

Subsequently we will be interested in rational coalitions functioning on the highest possible spectral level. It is obvious that the spectrum of each monotonic game on a finite set W is bounded, and therefore there exists a maximum spectral level $u^{\mu} = \max_{H \subseteq W} u[H]$.

Definition 3. *A rational coalition* V^* *such that for a certain choice* H^* *the level* u^{μ} : $u[H] = u^{\mu}$ *is attained is called the kernel of the monotonic parametric game on* W.

Theorem 1. *If* V_1^* *and* V_2^* *are kernels of the monotonic game on* W, *then one can always find the minimum kernel (in set-theoretic sense)* V_c^* *such that* $V_c^* \supseteq V_1^* \cup V_2^*$. *The proof is presented in the appendix.*

Theorem 1 asserts that the set of kernels in the sense indicated by the binary operation of coalitions is closed. The closeness of a system of kernels allows as looking at the largest (in the set-theoretic sense) kernel, i.e., a kernel $K^{\ominus}$ such that all other kernels are included in it. From the Theorem 1 it follows the existence of the largest kernel in any finite monotonic parametric game.

The rest of the paper is devoted to the description of constructive methods of setting up coalitions that are rational with the respect to the threshold u°, including those rational with the respect to the threshold u^{μ}, i.e., the kernels coalitions. In particular, a method of constructing the largest kernel is suggested.

7. SEARCH OF RATIONAL COALITIONS

We consider a monotonic parametric game with n players. Below we bring together a system of concepts, which allows us constructively to discover rational coalitions with respect to an arbitrary threshold u° if they exist. In the monotonic game only a limited portion of subsets of the set W have to be searched in order to discover the largest rational coalition. With this aim in the following we study coalitions V whose participants do not refuse from a choice: for $j \in V$ the choice $A^j \neq \varnothing$. Such a coalition, which has affected a choice H, is denoted by $V[H]$. From here on, for the motive of simplicity of notation of guaranteed utility $\pi_w(H \cap R_1, \dots, A^j, \dots, H \cap R_n)$, where H is a subset of the set W, we use $\pi(w; H)$.

Definition 4. *A sequence* $\overline{\alpha}$ *of elements* $\langle \alpha_0, \alpha_1, \ldots, \alpha_{m-1} \rangle$ *(*m *is the number of elements in* W *) from* W *is said to be in concord with respect to the threshold* u° *, if in a sequence of subsets of the set* W

$$\langle N_0, N_1, \ldots, N_{m-1}, N_m \rangle,$$

where $N_0 = W$, $N_{i+1} = N_i \setminus \alpha_i$, $N_m = \varnothing$, *there exists a subset* N_p *such that*:

a) *The utility* $\pi(\alpha_i; N_i) < u^\circ$ *for all* $i < p$;

b) *For each* $w \in N_p$ *the condition* $u^\circ \leq \pi(w; N_p)$ *is fulfilled, or, this being equivalent, for each* $j \in V(N_p)$ *the condition* $u^\circ \leq g_j(N_p)$ [11] *is fulfilled.*

A sequence $\overline{\alpha}$, in concord with the respect to the threshold u°, uniquely defines the set N_p. This fact is written in the form $N(\overline{\alpha}) = N_p$.

Definition 5. *A set* $S^\circ \subseteq W$ *is said to be in concord with the respect to a threshold* u°, *if there exists a sequence* $\overline{\alpha}$ *of elements of* W, *in concord with respect to the threshold* u° *and such that* $S^\circ = N(\overline{\alpha})$, *while the coalition* $V(S^\circ)$ *is said to be in concord with respect to the threshold* u°.

The following two statements are derived directly from Definitions 4 and 5.

A. In the case where the set $S^\circ = W$ is in concord with the respect to the threshold u°, all players $j \in I$ are not below u°: $g_j(W) \geq u^\circ$.

B. If the set S°, in concord with the respect to the threshold u°, is empty, then there exists a chain of constructing sets

$$\langle N_0, N_1, \ldots, N_{m-1}, N_m \rangle,$$

such that for each player $j \in I$, commencing with a certain N_t, in all those coalitions $V(N_i)$, $t \leq i$, where the player j enters, this player is below u°.

Theorem 2. *Let* S° *be a set that is in concord with respect to the threshold* u°. *Then any rational coalition* V *functioning on the level not less than* u° *makes a choice* H, *which is a subset of the set* S°: $H \subseteq S^\circ$. The proof is given in the appendix.

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[11] By definition $g_j(N_p) = \min_{w \in N_p \cap R_j} \pi(w; N_p)$.

Corollary 1. *The set* S°, *in concord with respect to the threshold* u°, *is unique*. Indeed, if we assume that there exists a set S', in concord with the respect to the threshold u° and different from S°, then from theorem 2, $S' \subseteq S^{\circ}$. But analogously at the same time the inverse inclusion $S' \supseteq S^{\circ}$ must also be satisfied, which bring us to conclusion that $S' = S^{\circ}$.

Corollary 2. *As the spectral levels of functioning of coalitions in the monotonic parametric game grow, one can always find a chain of rational coalitions, included in one another and being in concord with respect to each increasing spectral level, as with respect to the growing threshold.*

Indeed, from the formulation of the theorem it follows that a rational coalition, in concord with the respect to a spectral level $\lambda < \mu$, satisfies the relation $V(S^{\lambda}) \supseteq V(S^{\mu})$, since in a set-theoretic sense $S^{\lambda} \supset S^{\mu}$.

Below we arrange a certain sequence $\overline{\alpha}$, which use up all elements of W. After the construction we formulate a theorem about the sequence $\overline{\alpha}$ thus constructed being in concord with respect to the threshold u°. The arrangement proves constructively the existence of a sequence of elements of W that is necessary in the formulation of the theorem.

<u>Construction</u>. <u>Initial Step</u>.

<u>Stage 1</u>. We consider a set of elements W. Among this set we search out elements γ_0 such that $\pi(\gamma_0; W) < u^{\circ}$, and order them in any arbitrary manner in the form of a sequence $\overline{\gamma}_0$. If there are no such elements, then all elements of W are ordered arbitrarily in the form of a sequence $\overline{\alpha}$, and the construction is completed. In this case W is assumed to be the set $N(\overline{\alpha})$.

<u>Stage 2</u>. Subsequently we examine the sequence $\overline{\gamma}_0$. When considering the t-th element $\gamma_0(t)$ of this sequence $\overline{\gamma}_0$, the sequence $\overline{\alpha}$ is supplemented by the element $\gamma_0(t)$, which is denoted by the expression $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \gamma_0(t) \rangle$, while the set W is replaces by $W \setminus \overline{\alpha}$. After the last element of $\overline{\gamma}_0$ is examined we go over to the recursive step of the construction.

<u>Recursive Step</u> k.

<u>Stage 1</u>. Before constructions of the k-th step there is already composed a certain sequence $\overline{\alpha}$ of elements from W. Among the set $W \setminus \overline{\alpha}$ we seek out elements γ_k such that $\pi(\gamma_k; W \setminus \overline{\alpha}) < u^{\circ}$ and order them in any arbitrary manner in the form of a sequence $\overline{\gamma}_k$. Analogously to the initial step, if there happen to be no elements γ_k, the construction is ended. In this case in the role of the set $N(\overline{\alpha})$ we choose $W \setminus \overline{\alpha}$ while $\overline{\alpha}$ is completed in an arbitrary manner with all remaining elements from W.

Stage 2. Here we carry out constructions, which are analogous to stage 2 of the initial step. The entire sequence of elements $\overline{\gamma}_k$ is examined element by element. While examining the t-th element $\gamma_k(t)$ the sequence $\overline{\alpha}$ is complemented in accordance with the expression $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \gamma_k(t) \rangle$. After examining the last element $\gamma_k(t)$ of the sequences $\overline{\gamma}_k$ we return to stage 1 of the recursive step.

On a certain step p, either initial or recursive, at stage 1 there are no elements γ, which are required by the inequalities (2) or (3), and the construction could not continue any more.

Theorem 3. *A sequence $\overline{\alpha}$ constructed according to the rules of the procedure is in concord with the respect to the threshold* u°. The proof is presented in the appendix.

In the current section, in view of the use, as an example, of the concepts just introduced, we consider a particular case of a monotonic parametric game in which the difference in the individual and cooperative behavior of the participants of the coalition is easily revealed. We assume that the utilities

$$\pi_w(A^1,\dots,A^{j-1},X_j,A^{j+1},\dots,A^n)$$

do not depend on X_j in the case that choices specified by the remaining players are fixed. In this case the j-s participant of the coalition V, under the condition that the remaining participants of it keep their choices, can limit his choice X_j to a single element $w' \in R_j$ on which the maximum guarantee $\max_{w' \in R_j} g_j(H)$ is attained. However, such a selection narrowing his choice down to a single-element, generally speaking, reduces the choice (in view of monotonicity of utility indices π_w) to the guarantee of the remaining participants of the coalition. Consequently, individual behavior of the participants of a coalition contradicts their cooperative behavior. In spite of this contradiction, in the general case, in the given case, using the concept of a rational coalition $V(S^{\circ})$ in concord with respect to the threshold u°, and having slightly modified the criteria of "individual interests" of the players, we can convince someone that there always exists a situation in which the individual interests do not contradict the coalition interests.

We define the winnings of the j-th participant of the coalition in the form of the sum of utilities after subtraction of all payments u°, i.e., as the number

$$f_j(H) = \sum_{w \in A_j} [\pi(w;H) - u^{\circ}]$$

(the winnings f_k for $k \notin V$ are not defined). Having represented H as a joint choice $\left\langle A^1, A^2, \ldots, A^{|V|} \right\rangle$, we can consider the behavior of each j-th participant as player in a certain non-cooperative game selecting a strategy A^j.

The situation of individual equilibrium in the sense of Nash (Owen, 1971) of the participants of the coalition V in the game with winnings f_j is defined as their joint choice $\bigcup_{j \in V} A_*^j = H^*$ such that for each $j \in V$

$$f_j(A_*^1, \ldots, A_*^{j-1}, A^j, A_*^{j+1}, \ldots, A_*^{|V|}) \leq f_j(H^*)$$

for any $A^j \subseteq R_j$. In other word, the situation of equilibrium exists if none of the participants of the coalition has any sensible cause for altering his choice A_*^j under the condition that the rests keep to their choices.

Not every choice H of participants of the coalition V is an equilibrium situation. To see this it is sufficient to consider a choice H such that in the coalition V there are players having chosen elements $w \in A^j$ with utilities $\pi(w; H) < u^\circ$; for the selection of such an element the player pays more than this element brings in winnings $f_j(H)$ and, therefore, for the player, proceeding merely on the basis of individual interests, it would be advantageous to refrain from selection of such elements. Refraining from the selection of such elements of the set H is equivalent to non-equilibrium of H in the sense of Nash.

Lemma. *Let the utilities* $\pi(w; H)$ *be independent of* A^j. *Then a joint choice* S° *of the participants of the rational coalition* $V(S^\circ)$, *in concord with the respect to the threshold* u°, *is a situation of individual equilibrium.*

Indeed, according to Theorem 2, S° is the largest choice in the set-theoretic sense among all choices H of the rational coalition $V(S^\circ)$, where for any $w \in H$ the relation $\pi(w; H) \geq u^\circ$ is fulfilled. Let the choice of the participants of the coalition with an exception of that of the j-th participant be fixed. Since the utilities $\pi(w; S^\circ)$ do not depend on A^j, the j-th participant of $V(S^\circ)$ cannot secure an increase in the winnings $f_j(S^\circ)$ either by broadening or by narrowing his choice in comparison with $R_j \cap S^\circ$.

8. COALITIONS FUNCTIONING ON THE HIGHEST SPECTRAL LEVEL

We consider the problem of search of the largest kernel. First of all we present some facts, which are required for the solution of this problem.

From the definition of the guarantee $g_j(H)$ of the participant j effecting the choice H we see that the equality

$$g_j(H) = \min_{w \in A^j} \pi(w;H))$$

is fulfilled. Hence, according to the definition of the level $u[H]$ of functioning of the coalition $V(H)$ it follows that

$$u[H] = \min_{w \in H} \pi(w;H)$$

If we carry out a search of the subset H^* of the set W on which the value of the maximum of the function $u[H]$ is achieved, then thereby the search of a coalition functioning on the highest level $u^\mu = u[H]$ of the spectrum of a monotonic parametric game is affected. Without describing the search procedure, we give the definition of a sequence of elements W allowing us to discover the largest (in the set-theoretic sense) choice H^* of the largest coalition – a kernel K^*.

Definition 6. *A sequence* $\overline{\alpha}$ *of elements* $\langle \alpha_0, \alpha_1, ..., \alpha_{m-1} \rangle$ *(m is the number of elements in* W*) from* W *is called the defining sequence of the monotonic game, if in the sequence of sets* [12]

$$\langle N_0, N_1, ..., N_{m-1}, N_m \rangle$$

there exists a subsequence $\langle \Gamma_0, \Gamma_1, ..., \Gamma_p \rangle$ *such that:*

a) *for any element* $\alpha_i \in \Gamma_k \setminus \Gamma_{k+1}$ *of the sequence* $\overline{\alpha}$ *the utility* $\pi(\alpha_i; N_i) < u[\Gamma_{k+1}]$ $(k = 0,1,..., p-1)$;

b) *in the rational coalition* $V(\Gamma_p)$ *no sub-coalition exists on a level above* $u[\Gamma_p]$.

From the Definition 6 one can see that the defining sequence in many ways is analogous to a sequence, which is in concord with the respect to the level u°. Since any rational coalition $V(\Gamma_k)$ functions on the level $u^k = u[\Gamma_k]$, it is not difficult to note that the defining sequence $\overline{\alpha}$ composes strictly in-

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[12] The given sequence is constructed exactly in the same way as the one in Definition 4.

creasing spectral levels $u[\Gamma_0] < u[\Gamma_1] < ... < u[\Gamma_p]$ of functioning of rational coalitions $V(\Gamma_k)$ in the monotonic parametric game. As a result, we require yet another formulation.

Definition 7. *A rational coalition* $V \subseteq I$ *is said to be determinable, if there exists a defining sequence* $\overline{\alpha}$ *of elements* W *such that among the choices of this coalition there is a choice* Γ_p *composed by* $\overline{\alpha}$ *according to Definition 6.*

Theorem 4. *For each monotonic parametric game a determinable coalition exists and is unique. Among the choices of the determinable coalition there is a choice on which the highest spectral level* u^{μ} *is attained.* The proof of the theorem is presented in the appendix.

Corollary to Theorem 4. *The concepts of a determinable coalition and the largest kernel are equivalent.*

Indeed, directly from the formulation of the Theorem 4 we see that a determinable coalition always is the largest kernel. Hence, since a determinable coalition always exists, while the largest kernel is unique, it follows that the largest kernel coincides with the determinable coalition.

Thus, the problem of search of the largest kernel is solved if we construct a defining sequence $\overline{\alpha}$ of elements W. The construction of $\overline{\alpha}$ can be effected by the procedure of discovering kernels (KFP) from Mullat. In conclusion we present yet another approach to the concept of “stability” of a coalition. [13]

Definition 8. *A coalition* $\hat{V}$ *is said to be a critical, if for a certain choice* $\hat{H}$ *of it no coalition* V *having a nonempty intersection with the coalition* $\hat{V}$ *functions on a level higher than* $u[\hat{H}]$. *The level* $\hat{u} = u[\hat{H}]$ *is called the critical level of the coalition* $\hat{V}$, *while the choice* $\hat{H}$ *is called its critical choice.*

From the Definition 8, in particular, it follows at once the uniqueness of the critical level of the coalition $\hat{V}$. Indeed, on the contrary, if were two different levels $\hat{u}'$ and $\hat{u}''$, $\hat{u}' < \hat{u}''$, then $\hat{u}'$ could not be a critical one according to the definition: it is sufficient to consider the coalition $V = \hat{V}$ itself with the choice $\hat{H}''$, which ensures $\hat{u}'' > \hat{u}'$.

It is obvious that kernels are critical coalitions. The inverse statement, generally speaking, is not true; a critical coalition is not necessarily a kernel.

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[13] This approach is close to the concept of “M-stability” in cooperative n-person games, G. Owen.

We now consider the following hypothetical situation. Let $\hat{V}$ be a critical coalition and let $\hat{H}$ be its critical choice. We assume that this coalition is rational with respect to the threshold u°; i.e., $u^\circ \le u[\hat{H}]$ (see Definition 2). We assume that an increase of the threshold u° up to the level $u^\circ > u[\hat{H}]$ took place and the critical coalition $\hat{V}$ with the critical choice $\hat{H}$ was transformed into unstable coalition with respect to the higher threshold u°. Let the participants of the coalition $\hat{V}$ preserving the stability of the coalition attempt to increase their guarantees. One of the possibilities for increasing the guarantee of a participant $j_0 \in \hat{V}$ is to refrain from the choice of an element $\alpha_0 \in A^{j_0}$ on which the value $g_{j_0}(H)$ - the minimum level of utility guaranteed for him, see (4), is attained. It is natural to assume that a participant with a level of guarantee $g_{j_0}(\hat{H}) = u[\hat{H}] < u^\circ$ will be among the participants attempting to increase their guarantees, and refrains from the selection of the element α_0 indicated above. It may happen that the refusal of α_0 gives rise, for another participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, to a decrease from his guarantee $g_{j_1}(\hat{H}) > u[\hat{H}]$ to the quantity $g_{j_1}(\hat{H} \setminus \alpha_0) \le u[\hat{H}]$. A participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, acting from the same considerations as j_0, refrains from the selection of an element α_1 on which $g_{j_1}(\hat{H} \setminus \alpha_0)$ is attained. Such a refusal of α_1 can give rise to subsequent refusals, and emerges hereby a chain of "refusing" participants $\langle j_0, j_1, \ldots \rangle$ of the coalition $\hat{V}$.

If a coalition V, rational with respect to the threshold u° in the sense of Definition 2, with the choice H became unstable as the threshold u° increases, then such a coalition, generally speaking, disintegrates; i.e., some of its participants may become participants of a new coalition which already is rational with the respect to the increased threshold u°. By definition of a critical coalition, transaction of its participants into new rational coalition, when the threshold u° increases is not possible, and it disintegrates completely. The theorem presented below and proved in the appendix reflects a possible character of complete disintegration of a critical coalition in terms of the hypothetical system described above.

Theorem 5. *Let there be given a critical coalition* $\hat{V}$ *having a nonempty intersection with a certain coalition* V: $\hat{V} \cap V \neq \varnothing$. *Let* H *be the choice of the coalition* V *and* $\hat{H}$ *the critical choice of the coalition* $\hat{V}$. *Then in the coalition* $\hat{V} \cap V$ *there exists a sequence of its participants* $\bar{j} = \langle j_0, j_1, \dots, j_{r-1} \rangle$ *such that: a) in the sequence* $\bar{j}$ *there are represented all participants of the coalition* $\hat{V} \cap V$ *(the players* j_i *may be repeated,* r *is number of elements in* $\hat{H} \cup H$; *b) for the sequence* $\bar{j}$ *we can construct a chain of contracting coalitions*

$$\langle V(N_0), V(N_1), \dots, V(N_{r-1}) \rangle,$$

where $N_0 = \hat{H} \cup H$, $N_{i+1} \subset N_i$, *so that for any* $j \in V$, *commencing from a certain* N_t, *in all those coalitions* $V(N_i)$, $t \le i$, *into which the player* j *enters, this player is not above* $u[\hat{H}]$.

APPENDIX

Proof of Theorem 1. Let the level u^{μ} be attained for the coalitions V_1^* and V_2^*, which effect the choices H_1^* and H_2^* respectively; i.e., $u^{\mu} = u[H_1^*]$ and $u^{\mu} = u[H_2^*]$. For player $j \in I$ we consider two choices: $H_1^j = H_1^* \cap R_j$ and $H_2^j = H_2^* \cap R_j$ [14]. By the definition of guarantee $g_j(H_1^*)$ for the participant $j \in V_1^*$ of the coalition we have

$$\min_{w \in H_1^j} \pi_w (H_1^1, H_1^2, \dots, H_1^n) = g_j(H_1^*) \ge u^{\mu}; \qquad \text{(A1)}$$

for the participant $j \in V_2^*$ we respectively have

$$\min_{w \in H_2^j} \pi_w (H_2^1, H_2^2, \dots, H_2^n) = g_j(H_2^*) \ge u^{\mu}. \qquad \text{(A2)}$$

We determine the choice of a participant $j \in V_1^* \cup V_2^*$ as $\Phi^j = H_1^j \cup H_2^j$. The monotonic property (1) allows us to conclude that the following inequalities are valid:

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[14] We note that, in the worst case, for player $k \notin V_1^*$ ($k \notin V_2^*$), $H_1^k = \varnothing$ ($H_2^k = \varnothing$).

$$\min_{w\in H_1^j} \pi_w(\Phi^1,\Phi^2,...,\Phi^n) \geq \\ \geq \min_{w\in H_1^j} \pi_w(H_1^1, H_1^2,...,H_1^n) ; \quad (A3)$$

$$\min_{w\in H_2^j} \pi_w(\Phi^1,\Phi^2,...,\Phi^n) \geq \\ \geq \min_{w\in H_2^j} \pi_w(H_2^1, H_2^2,...,H_2^n) . \quad (A4)$$

Combining (A1) – (A4), we obtain

$$\min_{w\in\Phi^j} \pi_w(\Phi^1,\Phi^2,...,\Phi^n) \geq u^\mu \quad (A5)$$

for any $j \in V_1^* \cup V_2^*$. If by Φ^* we denote the set $H_1^* \cup H_2^*$, then for the coalition $V_1^* \cup V_2^*$ affecting the choice Φ^* the inequality (A5) is rewritten in the form

$$g_j(\Phi^*) \geq u^\mu,\ j \in V_1^* \cup V_2^* . \quad (A6)$$

Due to the monotonic property (1) some elements $w \notin \Phi^*$ (if one can find such) may be added to Φ^* while the inequality (A6) is still true [15]. We will denote the enlarged set by Φ^c: $\Phi^c \supseteq \Phi^*$ and obviously for $V^c = V(\Phi^c)$ we have $V(\Phi^c) \supseteq V_1^* \cup V_2^*$. By the definition of a spectral level u^μ, for the participant $j' \in V^c$, on which $u[\Phi^c]$ is attained, we have

$$g_{j'}(\Phi^c) = u[\Phi^c] \leq u^\mu , \quad (A7)$$

since u^μ is the maximum spectral level of functioning of coalitions in the monotonic game. Applying (A7) and (A6) to the choice Φ^c for the participant $j = j'$, we see that $g_{j'}(\Phi^c) = u^\mu$, and the coalition $V^c \supseteq V_1^* \cup V_2^*$ functions on the spectral level u^μ. The theorem is proved. ■

Proof of Theorem 2. Let S° is a subset of the set W in concord with the respect to the threshold u°; i.e., there exists a sequence $\overline{\alpha}$, in concord with the respect to the threshold u°, such that $S^\circ = N(\overline{\alpha})$. We assume that there exists a coalition V affecting a choice $H \subset S^\circ$ and functioning on the level

[15] We suppose that such elements cannot be added to Φ^c.

$u[H] \geq u^\circ$, $H \setminus S^\circ \neq \varnothing$. Let $\alpha_1 \in H \setminus S^\circ$ and let α_t be an element, which is leftmost in the sequence $\overline{\alpha}$. Let p be the index of the set N_p in the sequence $\langle N_0, N_1, \ldots, N_{m-1}, N_m \rangle$. It is obvious that $t < p$ and, consequently,

$$\pi(\alpha_t; N_t) < u^\circ \tag{A8}$$

in accordance with a) of the Definition 4. Since the game being considered is monotonic, $\alpha_t \in H$ and $H \subseteq N_t$ there must hold

$$\pi(\alpha_t; H) \leq \pi(\alpha_t; N_t). \tag{A9}$$

From inequalities (A8) and (A9) it follows

$$\pi(\alpha_t; N_t) < u^\circ \leq u[H] \tag{A10}$$

(the latter $\leq$ by assumption). According to the inequality (A10) and by the definition of $u[H]$ we have

$$\pi(\alpha_t; H) < \min_{j \in V} g_j(H). \tag{A11}$$

Let the element α_t be chosen by a certain q-th player; i.e., $\alpha_t \in A^q$, $q \in V$. On the basis of (A11) we assume that

$$\pi(\alpha_t; H) < g_q(H) \tag{A12}$$

is valid. By definition $g_q(H) = \min_{w \in A^q} \pi(w; H)$ and following (A12), we note that $\pi(\alpha_t; H) < \min_{w \in A^q} \pi(w; H)$. The last inequality is contradictory, what proves the theorem. ■

Proof of Theorem 3. We assume that the construction of the sequence $\overline{\alpha}$ according to the rules of the procedure ended on a certain p-th step. This means that $\overline{\alpha}$ is made up of sequences $\overline{\gamma}_k$ $(k = \overline{0, p})$, and also of elements of the set N_p, found according to the rules of the procedure and being certainties for the sequences $\overline{\gamma}_k$. We consider any element α_i of the sequence thus constructed, being located on the left of the α-th element: $i < p$. The given element in the construction process falls into certain set $\overline{\gamma}_q$. By construction

$$\pi(\alpha_i; W \setminus \{\overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \ldots \cup \overline{\gamma}_{q-1}\} < u^\circ. \tag{A13}$$

If to the sequence $\langle \overline{\gamma}_0, \overline{\gamma}_1, \ldots, \overline{\gamma}_{q-1} \rangle$ we add the elements $\overline{\gamma}_q$, which in $\overline{\alpha}$ are on the left of the α_i-th, then this set of elements together with the added part $\overline{\gamma}_q$ composes the complement $\overline{N}_i$ up to the set W (see Definition 4).

On the basis of the monotonic property (1) we conclude that $\pi(\alpha_i; W \setminus \{\overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \ldots \cup \overline{\gamma}_{q-1}\} \geq \pi(\alpha_i; W \setminus \overline{N}_i) = \pi(\alpha_i; N_i)$. The last relation in the combination with (A13) shows that $\pi(\alpha_i N_i) < u^\circ$. From the construction of the sequence $\overline{\alpha}$ it is also obvious that for any $j \in V(N_p)$ the guarantee $g_j(N_p) \geq u^\circ$. The theorem is proved. ■

Proof of the Theorem 4. Theorem can be proved as follows. First, a sequence $\overline{\alpha}$, in concord with respect to the highest spectral level u^μ, in the monotonic game exists, according to Theorem 3, and is, at the same time, a defining sequence; as the subsequence $\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \rangle$ in this case we have to choose the sequence $\langle W, S^\mu \rangle$, where S^μ is a set $S^\mu \subset W$ which is in concord with respect to the highest level u^μ. The determinable coalition is $V(S^\mu)$. The uniqueness of the coalition $V(S^\mu)$ is proved in Corollary 1 to the Theorem 1. Secondly, the choice S^μ of the coalition $V(S^\mu)$, playing the part of the set Γ_p in the Definition 6, attains the maximum of the function $u[H]$, a fact which follows from Theorem 3 and b) of Definition 6; i.e., $u[S^\mu] = u^\mu$. Thirdly, the last statement of Theorem 4 is a particular case of the statement of Theorem 2, if we put $u^\circ = u^\mu$. The theorem is proved. ■

Proof of the Theorem 5. We consider a monotonic game of participants of a coalition $\hat{V} \cup V$ on the set $\hat{H} \cup H$, where $\hat{H}$ is the critical choice of the critical coalition $\hat{V}$, and H is some choice of the coalition V. Below the set $\hat{H} \cup H$ is denoted by Ω, while all concepts refer to a monotonic sub-game on Ω.

Let u° be the threshold of the parameter u of the game on Ω, and let $u^\circ > u[H]$. We construct a sequence $\overline{\alpha}$ of elements Ω, which is in concord with respect to the threshold u°. Two variants could be represented: 1) the set S°, in concord with the respect to the threshold u° is empty; 2) S° is not empty. We consider them one after the other. First, in the variant 1) from a sequence of elements $\overline{\alpha}$ of elements of Ω in concord with respect to the threshold u°, we uniquely determine a sequence of participants of the coalition $\hat{V} \cup V$ choosing elements α_i from sequence $\overline{\alpha}$ and composing a certain chain $\overline{j} = \langle j_0, j_1, \ldots, j_{r-1} \rangle$ (r is the number of elements Ω). Secondly, from the sequence $\overline{\alpha}$ we also uniquely determine the sequence of coalitions $\langle V(N_0), V(N_1), \ldots, V(N_{r-1}) \rangle$, where $N_0 = \Omega$, $N_{i+1} = N_i \setminus \alpha_i$, with $j_i \in V(N_i)$.

In the second variant none of the participants of the coalition $\overline{V}$ can be in a coalition, which is in concord with the respect to the threshold $u^{\circ} > u[H]$. This would contradict the definition of a critical coalition $\overline{V}$. Therefore in the chain $\overline{j}$ thus constructed of participants of the coalition $\hat{V} \cup V$ (by the same method as in the first variant) all participants of the coalition $\overline{V}$ are on the left of the j_p-th player; p is uniquely determined from the sequence $\overline{\alpha}$ (see Definition 4). By property a) of the Definition 4 and from the definition of the guarantee of a player $j_i \in V(N_i)$ we have

$$g_{j_i}(N_i) \le \pi(\alpha_i; N_i) < u^{\circ}. \qquad \text{(A14)}$$

Proceeding from the structure of the spectrum of a monotonic parametric game on Ω (see Corollary 2 to the Theorem 2) the value u° marginally close to $u[H]$ is satisfied successfully in the two variants considered. The first variant of the Theorem 5 forms the statement b) derived earlier from Definition 4 and 5 (see section 2). The second variant of the statement of the theorem is directly derived from the relation (A14). ■

References

Algaba, E., Bilbao, J. M., van den Brink R. and A. Jiménez-Losadaa. (2004) Cooperative games on antimatroids, Discrete Mathematics, 282, pp. 1-15.

Joskow, P. L. (2005) Patterns of Transmission Investment, Paris, Leveques Conference, revised draft,.

Knuth, D. (1972) The Art of Computer Programming, Vol.1: Fundamental Algorithms, Addison-Wesley, Reading, Massachusetts.

Korte, B., L. Lov´asz and R. Schrader. (1991) Greedoids, Springer-Verlag, New York/Berlin.

Levit, V. E. and Y. Kempner. (2008) Quasi-concave functions on antimatroids, Department of Computer Science, Holon Academic Institute of Technology, Holon 58102, Israel.

Mullat, J. E. a) (1979) Stable Coalitions in Monotonic Games, Translated from Avtomatica i Telemekhanika No. 10, pp. 84 – 94. Original article submitted 1978. Automation and Remote Control, Plenum Publishing Corporation 1980, pp. 1469-1478, b) (1980) Equilibrium in a Production Network, Translated from Avtomatica i Telemekhanika, 12, pp. 124 – 131. Original article submitted 1979. Automation and Remote Control, Plenum Publishing Corporation 1981, pp. 1724-1729; c) Extremal Subsystems of Monotonic Systems, I,II,III," *Automation and Remote Control, 1976, 37*, 758-766, 1976, *37*, pp. 1286-1294; 1977, *38*. pp. 89-96.

Nash, J. F. (1953) Two-Person Cooperative Games, Econometrica 21, pp. 128-140.

Owen G. a) (1968) Game Theory, Saunders Philadelphia; b) (1982) Game Theory (Russian translation) Mir. Second Edition New York London, Academic Press, INC. (LONDON).

Rapoport, A. (1983) Mathematical Models in the Social and Behavioural Sciences, Copyright ©, John Wiley & Sons.

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[i] In his book review of "Ménard, C. and M.M. Shirley. (eds., 2005) *Handbook of New Institutional Economics*, Springer: Dordrecht, Berlin, Heidelberg, New York. XIII. 884 pp., Rudolf Richer, University of Saarland, noticed that

> *North and Williamson stress, besides transaction costs, the role of bounded rationality, uncertainty, and imperfect rationality. Their objects of research differ: Northian NIE focuses on macro institutions that shape the functioning of markets, firms, and other modes of organizations such as the state (section II) and the legal system (section III). Williamsonian NIE concentrates on the micro institutions that govern firms (section IV), their contractual arrangements (section V), and issues of public regulation (section VI). Both the Northian and Williamsonian approaches to the NIE are used, i.e., in development and transformation economics: in efforts towards explaining the differences of exchange-supporting institutions (section VIII).*

It is worth to emphasize, in view of the above, that when the player $j \in V$ must make a payment u° for the element $w \in A^{j}$, the payment is well suited in the role of transaction cost, see below.

> *In economics and related disciplines,* ***a transaction cost*** *is a cost incurred in making an economic exchange. For example, most people, when buying or selling a stock, must pay a commission to their broker; that commission is a transaction cost of doing the stock deal. Or consider buying a banana from a store; to purchase the banana, your costs will be not only the price of the banana itself, but also the energy and effort it requires to find out which of the various banana products you prefer, where to get them and at what price, the cost of travelling from your house to the store and back, the time waiting in line, and the effort of the paying itself; the costs above and beyond the cost of the banana are the transaction costs. When rationally evaluating a potential transaction, it is important to consider transaction costs that might prove significant.*

FINAL REMARKS

It ends where we started. The paper investigated a situation of distributing commodities in the retail chain with participants making "*to buy and sales*" decisions in a retail chain. One type of participants' produce and sale, others buy and sale, the third only buy for consumption. The price system was set up via some constants, which are nothing but percentages to perform calculus of how the sales price must depend and exceed the purchasing prices to archive a satisfactory results for participants maximizing their profits. The situation becomes complex as soon as to buy and sale decisions incorporated transaction costs. Transaction costs interact into the behavior of participants by transforming potentially profitable into loosing transactions. The paper investigated the situation, as global, depending on the transaction costs' threshold varying the threshold from low to high values until all transactions, allegedly profitable in bilateral trade agreements, became loosing and do not any more form a basis for an agreement between rational participants. The retail chain structure, while the transaction costs' threshold is increasing, changes like nested set of retail chains each of them on the higher threshold is capable to counteract higher transaction costs and still functioning in equilibrium. Condition for such a rational behavior was that all participants in the retail chain must avoid any loosing transaction. Beyond the goal of the retail chain formation to hold the retail chain in equilibrium, some elasticity intervals for transaction costs, where it still was realistic to buy and sale rationally, have been internally encoded into the scheme and calculated individually for all participants in the chain.

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The Section 5 is a rewritten and expanded version of an article that appeared in "Automation and Remote Control", ISSN 005-1179 (Mullat, 1980, p.7), Russian periodical of MAIK Наука, Интерпериодика. In the expanded version, the introduction reflects the relationship with the new institutional economics, while final remarks point at elements of the theory of firm formation, as it has now become clear to the author. Nevertheless, significant amendments to the terminology have also been made in the main version of the article in English, which was originally distributed by Plenum Publishing Corporation.