Wellness Club/Coalition Formation by Bargaining on Boolean Tables

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Abstract. The concept or category of bargaining according to the Nash Bargaining Problem is associated with the game of club/coalition formation using the derivation of utility functions based on Boolean tables. It will appeal to specialists in the social sciences and economics.

Key words: coalition; game; bargaining; algorithm; monotonic system

1. Introduction

Science differs from any other sphere of human activity in its goals, means, motives and conditions in which scientific work proceeds (Popper, 2002; Ponterotto, 2005; Abhary et al., 2009). If the goal of science is the comprehension of truth, then its procedure is scientific theory. Theory, unlike spontaneous forms of cognition of the surrounding world, is based on the norm of the activities of acquiring knowledge – the scientific method. Its implementation involves understanding and consolidating the theory. This means methodology, approaches, techniques, application orientation and reproducibility of results.

Usually, a theoretical or practical contribution to the theory or practice of applying a theory consists in expanding existing categories, concepts, models, simplifications, etc. towards obtaining new theoretical facts or solving unsolved problems. However, there is another approach to extracting new knowledge from old and well-established categories, which consists in recognizing new relationships or links hidden between old fundamental categories. This innovation taking two things that already exist and putting them together in a new way is the main motive of this article – a comparison, or rather, an interpretation of the well-known Nash Bargaining scheme with the theoretical provisions of the coalition game as applied to Boolean tables. The application of this provision lies in the fact that Boolean tables allow you to calculate the utility functions of the coalition game and thereby determine the individual division of the total payoff or revenue for each player separately. For this purpose, we have developed an example of a game in a popular “wellness club" language of a barmaid to illustrate what such an innovation can do in terms of Boolean table matching the Bargaining Problem and coalition games.

Therefore, the specific type of game situation on Boolean tables, as it seems to us, does not limit, but rather enriches the theory and provides additional tools, which in practice affect socio-economic stability and open up opportunities for data analysis, both in social networks and for the interpretation of rational behavior of participants in network models in the economy.

In conclusion, we were also able to illustrate various utility functions of coalition games (so-called super-modular/concave) that are actually responsible for the coalition motivation of players when receiving payouts (super-modular functions) or when the collective behavior of the players loses its appeal (super-concave functions).

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Since the publication of “The bargaining problem” by John F. Nash, Jr. in 1950, the framework proposed within has been developed in different directions. For example, in their “Bargaining and Markets” monograph, Martin Osborn and Ariel Rubinstein (1990) extended the “axiomatic” concept initially developed by Nash to incorporate a “strategic” bargaining process pertinent to everyday life. The authors posited that the “time shortage” is the major factor encouraging agreement between bargainers. Various bargaining problem varieties emerged in the decades following Nash’s pioneering work, prompting game theoreticians to seek their solutions, most of which did not necessarily comply with all Nash axioms. Beyond any doubt, the “Nonsymmetrical Solution” proposed by Kalai (1977); Harsanyi’s (1967) “Bargaining under Incomplete Information”; “Experimental Bargaining”, which was later proposed by Roth (1985); and the “Bargaining and Coalition” paper published by Hart (1985) are among some notable contributions to this field, confirming the fundamental importance of bargaining theory.

Bargaining and rational choice mechanisms are interrelated concepts and are treated as such in this work. In the context of general choice theory, the choice act can be formalized through internal and external descriptions, which requires use of binary relations and the theoretical approach, respectively. Thus, both description modes apply to the same object, albeit from different perspectives. The Nash Bargaining Problem and its solution express exactly the same phenomenon. Given a list of axioms, such as “Pareto Efficiency” or “Independence of Irrelevant Alternatives”, in terms of binary relations the rational actors must follow, the solution is reached through scalar optimization applied to the set of alternatives. Indeed, the scalar optimization is at the core of the Nash’s axiomatic approach and is the reason for its success in performing the bargaining solution derivation. In this respect, the motive of this work is also to present a “derivation” of bargaining solution on large Boolean Tables and some theoretical foundations offered by the method. Unfortunately, in following Nash’s scenario, numerous difficulties emerged.

Boolean Table representation transforms the "cacophonous" of real-life scenario into a relatively simple scalar index, characterizing an understandable data format (Malik and Zhang, 2009). However, given the ambiguity of scalar optimization, makes the picture more complicated. We are considering a purely atomic object that does not intuitively satisfy the “invariance under the change of scale of utilities” postulate typically assumed in the proofs. From the researcher’s point of view, the issue stems from the incertitude pertaining to the most optimal choice of the scalar criteria. The Nash axiomatic approach suggests that employing the product of utilities is the most appropriate, thus removing any uncertainty from further discussion. Nevertheless, in the context of the method presented here, it is posited that a reasonable solution might come into consideration, while game-analysts would be advised to include the method in a wider range of applicable game analysis tools.

The following section provides a basic example of our bargaining game. In the appendix, we also illustrate another negotiation scheme between a coalition and its manager on Boolean tables using some of the usual utility functions. It is worth noting that some elements of the main example, such as signals or distortions, are not the main topic of our discussion. These points should rather be understood as illustrating the complexity of the negotiation process.

In Section 3, we attempt to explain our intentions in a more rigorous manner. Here, we formulate our "Bargaining Problem on Boolean Tables" in pure strategies, thus providing the foundation for Section 4, where we exploit our pure Pareto frontier in terms of so-called Monotonic Systems chain-nested alternatives – the Frontier Theorem. In order to implement the Nash theorem for nonsymmetrical solution (Kalai, 1977), in Section 5, we introduce what we deem to be an acceptable, albeit complex, algorithm in general form. Even though lottery is not permitted in the treatment of Boolean Tables subsets representing pure strategies, as this approach does not necessary produce the typical convex collection of feasible alternatives, we claim that the algorithm will yield an acceptable solution. Finally, Section 6 presents an elementary attempt to formulate a regular approach of coalition formation under the coalition formation supervisor – the manager's structure. This attempt depicted in Figure 2, explaining the notation nomenclature of chain-nested alternatives adopted in our Monotonic Systems theory, discussed in Section 4. Section 7 summarizes and discuses the entire analysis, while also providing an independent heuristic interpretation, before concluding the study in Section 8.

Concise Glossary of Mathematical Notations

Matrix $W = \left\| \alpha_{ij} \right\|_n^m$ signifies the Boolean table, $\alpha_{ij} = 1$ or $\alpha_{ij} = 0$, which denote the Boolean element of the table W. For players joint expectations we use the notation (x, y), where $x \in 2^N, N = \{1,\ldots,i,\ldots,n\}, x$ – subset of rows N, and $y \in 2^M, M = \{1,\ldots,j,\ldots,m\}, y$ – subset of columns. Sub-table H or block is denoting the players' joint crossing of rows x and columns y, Notation $|H|$ indicates the number $\sum_{\alpha_{i\,j \in H}} \alpha_{ij}$ of 1's in the sub-table H. Pairwise operations – $L \cup G$, $L \cap G$, $H = \varnothing$, $L \subset G$, $\supset$, $L \subseteq G$, $\supseteq$, $H \subseteq W$, $W \supseteq \Gamma$. We sometime use the notation $H + i$, $H + j$ and similarly for $H \setminus i$, $H \setminus j$ we use $H - i$, $H - j$ for $i \in x$, $j \in y$ and so for $H \cup i$, $H \cup j$. The notation $\langle \alpha_1, \alpha_2, \ldots \rangle$ indicates to the ordered set of elements while $\{\alpha_1, \alpha_2, \ldots\}$ represents unordered set.

2. Wellness Game on Boolean Table

The Chief Executive Officer of the "Well-Being" company encouraging employees to partake in wellness-promoting events or activities is determined to reduce company losses arising from disability compensations. To identify the employees' preferences, the CEO has initiated a survey. According to the survey responses, five wellness events offered to the employees generated varying degrees of interest, as shown in Table 1.

Wellness events	*No Smoking*	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 1		x	x			2
Em. No. 2	x	x		x	x	4
Em. No. 3		x	x	x		3
Em. No. 4	x	x		x	x	4
Em. No. 5	**Heavy smoker**	**Clumsy swimmer**	x	x		2
Em. No. 6	x	x	x	x	x	5
Em. No. 7		x	x			2
Total	3	6	5	5	3	22

Table 1 Employee preferences pertaining to the company-sponsored wellness-promoting events

The CEO would like to consider staff responses as an indication that they are ready to participate in their chosen events. Knowing the precariousness of human nature, CEO is not sure that they will keep their promises. That's why the CEO decides to reward all employees who actually participate in recreational events, and those who will be organized in the "Wellness Club". The CEO found a sponsor who issued 12 banknotes to cover the cost of the project. Upon closer examination of the rewords policy, the CEO realized that many obstacles had to be overcome in order to put the project into practice.

First, organizing events that only a few employees would partake in is neither practical nor cost-effective. Indeed, it is necessary to stipulate a minimum number of employees that must subscribe to each wellness event. On the other hand, it is desirable to promote all events, encouraging the employees to attend them in greater numbers. For this initiative to be effective, instructions (as a rule full of twists and turns) regarding the rewards regulations should be fair and concise. Usually, in such situations, someone (a manager) must be in charge of the club formation and reward allocation. As the CEO is responsible for financing the wellness events, he/she should retain control of all processes. Thus, the CEO proposes to write down the **First Club Regulation**: *The CEO rewards one Bank Note to an employee participating in at least* k *different events* (where k is determined by the CEO).

Determining the most optimal value of the parameter k is not a straightforward task, as it is not strictly driven by employees' preferences regarding specific events to participate in. In fact, this task is in the manager's jurisdiction, while also being dependent on the employees' decisions, as they act as the club members. The goal is to prohibit some club members to "spring over" wellness events preferred by other members of the club by worsening, in the CEO's view, the situation, thus requiring too many different events to be organized. This issue can be avoided by the inclusion of the **Second Club Regulation**: *If a member of the club being organized expresses an intention to participate in less than* k *events in favor of receiving a reward, none of the members of the future club is rewarded.* By instituting this regulation, the CEO aims to encourage the manager to eliminate events that would not have sufficient number of participants. Thus, the **Third Club Regulation**: *manager's reward basket will be equal to the lowest number of participants per event in the list of events among all actually participating club members.* Indeed, to earn more rewards, the manager might decide to organize a new club by excluding an event with the lowest number of participants from the list of events some of the members chose to attend as a part of the already organized club. This would effectively result in the lowest number of participants in the new and shorter list being higher than that in the previous list. It should also be noted that the third reward regulation does not address the situation in which a club member declines an event, allowing an individual outside the club to participate instead. In such a case, the club "events list" may become shorter than that presented in Table 1, and would determine the size of the manager's reward.

This scenario also provides the potential for the club members' preferences to be misrepresented to the company CEO. Let us assume that the CEO makes a decision $k=1$, which has been, for whatever reason, made accessible to the manager. Knowing that $k=1$, the manager actions can be easily predicted in accordance with the third club regulation. Indeed, using the employees' survey responses, the manager can identify the most "popular" wellness event, as well as the individuals that intend to participate in this event. It can be seen from the above provision that the manager will receive the maximum reword if the latter succeeds in convincing other employees from participating in any other event, but only in this one particular event. Rational club members would certainly agree to that proposal because, whether or not they take part in any other event, their reward is still guaranteed.[2] The same logic obviously applies for $k>1$ as well.

2 We will disclose more complex misrepresentation opportunity later.

The essence of establishing fair rules is related to the determination of the manager's leadership. If no rewards are offered to the manager, the formation of a grand coalition is guaranteed, as all employees will become members of the club. This is so because participating in any event guarantees that all employees will receive a reward. However, due to the lack of interest in events with a small number of participants, the formation of a club with a large number of participants under the leadership of a manager is not always feasible.

As previously noted, the manager might receive a minor reward if a "curious" employee decides to take part in an "unpopular event". Indeed, the third club regulation stipulates that the number of participants in the most "unpopular event" governs the manager reward size. Being aware of the potential manipulation of the regulations, and being a rational actor, the company CEO will thus strive to keep the decision k in a secret. It is also reasonable to believe that all parties involved – the club members, the manager and the CEO – will have their own preferences regarding the value of k. Therefore, an explanation based on the salon game principles is applicable to this scenario. Using this analogy, let us assume that the CEO has chosen a card k and has hidden it from the remaining players. Let us also assume that the manager and the club members have reached an agreement on their own card choice in line with the three aforementioned club regulations. The game terminates and rewards are paid out only if their chosen card is higher than that selected by the CEO. Otherwise, no rewards will be paid out, despite taking into consideration the club formation.

Not all factors affecting the outcome have been considered above. Indeed, the positive effect, f_k, which the CEO hopes to achieve, depends on the decision k. We have to expect a single $\cap$-peakedness of the effect function for some reason. As a result, this function separates the region of k values into what we call prohibitive and normal range. In the prohibitive range, which includes the low k values, the effect has not yet reached its maximum value. On the other hand, when k value is high (i.e., in the normal range), the f_k limit is exceeded. Therefore, in the prohibitive range, the CEO and the manager interests compete with each other, making it reasonable to assume that the CEO would keep his/her decision a secret. In the normal range, they might cooperate, as neither benefits from very high k values, given that both can lose their payoffs. Consequently, using the previous card game analogy, in the normal range, it is not in the CEO's best interests to hide the k card.

Given the arguments presented above, the game scenario can be illustrated more precisely. Using the data presented in Table 1, and assuming that a reward will be granted at $k = 1,2$, the CEO may count upon all seven employees to become the club members. If all employees participate in all events, each would receive a Bank Note, and the manager's basket size would be equal to 3.

It would be beneficial for the manager to entice to the club members to decline participation in *"No Smoking"* and *"Fattening Diet"* events, as this would increase his/her own reward to 5. As all club members will still preserve their rewards, they have no reason not to support the manager's suggestion, as shown in Table 2.

Table 2

Wellness events	Swimming Pool	Fitness Exercises	Moderate Alcohol	Total
Em. No. 1	x	x		2
Em. No. 2	x		x	2
Em. No. 3	x	x	x	3
Em. No. 4	x		x	2
Em. No. 5		x	x	2
Em. No. 6	x	x	x	3
Em. No. 7	x	x		2
Total	6	5	5	16

Table 3

Swimming Pool	Total
x	1
x	1
x	1
x	1
	0
x	1
x	1
6	6

In this scenario, the sponsor would have to issue 12 Bank Notes, which can be treated as expenses associated with organizing the club. The sponsor may also conclude that $k = 1$ is undesirable based on the previous observation that the manager can deliberately misrepresent the members' preferences for personal gain.[3] Indeed, the manager can offer one Bank Note to the employee when the CEO makes a decision $k = 1$. Knowing that $k = 1$, the manager may suggest to the club members to subscribe to the *"Swimming Pool"* only. However, in opinion of the potential swimming club, the manager must compensate the heavy smoker and clumsy swimmer No. 5 for the losses sustained. Employee No. 5, participating, e.g., in no smoking and no swimming clubs, is otherwise entitled to receive compensation and may signal the fraud of the manager to lobbyists of the company's board: No. 5 will be able, while continuing to smoke, to demand compensation from the manager for not disclosing the latter's "activities". In this case, following the regulations in force (see Table 3), manager's reward will be equal to 4 (1 deducted for the signal and 1 for clumsy swimmer). This would still exceed the value indicated in Table 1. Thus, in order to decrease project expenses or avoid misrepresentations, the company board may follow the swimming club advice and propose $k \geq 3$.

It could be argued that $k \geq 3$ results in decreased participation in wellness events because Employees No. 1, 5 and 7 will be excluded from the club and will immediately cease to partake in any of their initially chosen events. Based on Table 4, it can also be noted that, in such an event, the remaining employees (i.e., $2,3,4$ and 6) will still participate in heath events and will still be rewarded.

3 The more complex case of misrepresentation follows, as promised.

Table 4

Wellness events	*No Smoking*	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 2	x	x		x	x	4
Em. No. 3		x	x	x		3
Em. No. 4	x	x		x	x	4
Em. No. 6	x	x	x	x	x	5
Total	3	4	2	4	3	16

Now, the manager's reward basket is equal to 2, since only Employees No. 3 and 6 would take part in *"Fitness Exercises"*. Consequently, the sponsor expenses decrease from 10 to 6. In this case, the CEO may decide to allow the manager to retain his/her reward of 3 by eliminating *"Fitness Exercises"* from the event list, as organizing it for two participants only is not justified, as shown in Table 5. Note that Employee No. 3, due to this decision, must be excluded from the club list, in line with the second club regulation, cf. the suggestion above to eliminate *"No Smoking"* and *"Fattening Diet"* events.

Table 5

Wellness events	*No Smoking*	*Swimming Pool*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 2	x	x	x	x	4
Em. No. 4	x	x	x	x	4
Em. No. 6	x	x	x	x	4
Total	3	3	3	3	12

This decision does not seem reasonable, given that the aim of the initiative was to motivate the employees to fitness exercise and improve their wellness. Thus, let us assume that $\mathrm{k}=5$ was the board proposal. This result would only concern Employee No. 6 being willing to participate in the wellness events offered, see Table 6.

Table 6

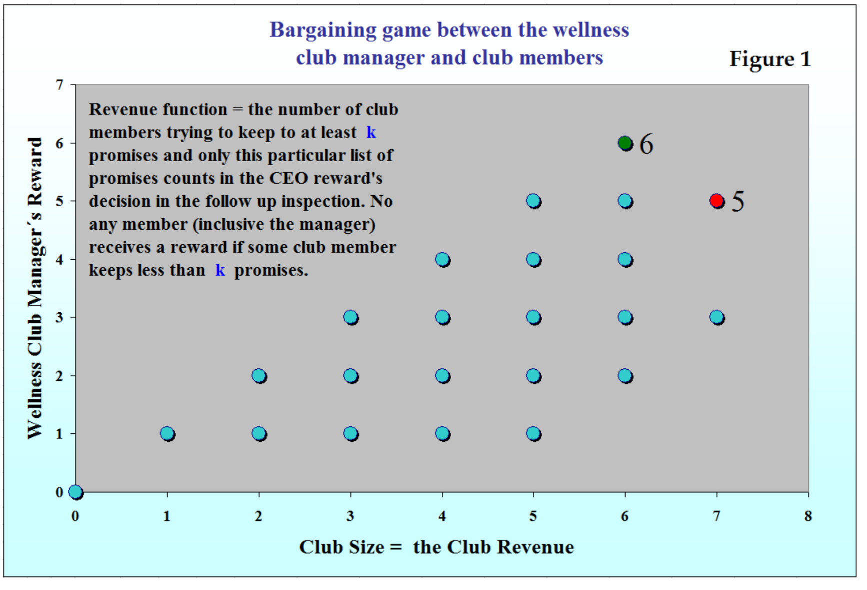
Wellness events	*No Smoking*	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 6	x	x	x	x	x	5
Total	1	1	1	1	1	5

The manager may decide not to organize the club, as this would result in a reward equal to only one Bank Note. Similarly, the CEO is not incentives to promote all five events if only one employee would take part in each one. As a result, at the board meeting, the CEO would vote against the proposal $\mathrm{k}=5$. In sum, the CEO's dilemma pertains to the alternative k choice based on the information given in Table 7.

Table 7

	Club members	*Club manager*	*Compen-sation*	*Signal*	*Bank Notes used*	*Bank Notes left*
T. 1, $k=2$	7	3	0	0	10	2
T. 2, $k=2$	7	5	0	0	12	0
T. 3, $k=1$	6	4	1	1	12	0
T. 4, $k=4$	3	1	0	0	4	8
T. 5, $k=4$	3	3	0	0	6	6
T. 6, $k=5$	1	1	0	0	2	10

To clarify the situation presented in tabular form, it would be helpful to visualize the CEO's dilemma using the bargaining game analogy, where 12 Bank Notes are shared between the manager and the club members.



The decision on the most optimal k value taken at the board meeting will be revealed later, using rigorous nomenclature, as only a closing topic is necessary to interrupt our pleasant story for a moment.[4]

Let us assume that three actors are engaged in the bargaining game: N employees, a manager in charge of club formation, and the CEO. Certain employees from $N = \{1,\ldots,i,\ldots n\}$ – the potential members of the club x, $x \in 2^N$, have expressed their willingness to participate in events y,

[4] Those unwilling to continue with the discussions on bargaining presented in the subsequent sections should nonetheless pay attention to this closing remark.

$y \in 2^M$, $W = \| \alpha_{ij} \|_n^m$. Let a Boolean Table $W = \| \alpha_{ij} \|_n^m$ reflect the survey results pertaining to employees' preferences, whereby $a_{ij} = 1$ if employee i has promised to participate in event j, and $a_{ij} = 0$ otherwise. In The set 2^M of all subsets of columns M denotes of allegedly subsidized events, whereby $y \in 2^M$ have been examined: $M = \{1,..., j,...m\}$.

We can calculate the manager payoff $F_k(H)$ using a sub-table H formed by crossing entries of the rows x and columns y in the original table W by further selection of a column with the least number $F_k(H)$ from the list y. The number of 1-entries in each column belonging to y determines the payoff $F_k(H)$. Utility functions family $\upsilon^k(x, y) \equiv \upsilon^k(H)$, $k \in \{1,...,k,...,k_{max}\}$, on N are known for the coalition games; in particular, for every pair $L \subset G$, $L, G \in 2^N \times 2^M$, we suppose that $\upsilon^k(L) \le \upsilon^k(G)$. Further assuming that the CEO payoff function $f_k(H)$ has a single $\cap$-peakedness, in line with the decisions $\langle 1,...,k,...,k_{max} \rangle$, $f_k(H)$ reflects some kind of positive effect on the company deeds. In this case, sponsor expenses will be equal to $\upsilon^k(H) + f_k(H)$.

Finally, it is appropriate to share some ideas regarding a reasonable solution of our game. The situation is similar to the Nash Bargaining Problem first introduced in 1950, where two partners – the club members and the manager – are striving to reach a fair agreement. It is possible to find the Bargaining Solution $S_k \in \{H\} = 2^N \times 2^M$ for each particular decision k, see next sections. The choice of the number k is not straightforward, as previously discussed. For example, $k = 4, 5$ may be useful based on some *ex ante* reasoning, whereas maximum payoffs are guaranteed for the partners when $k = 1$. As that decision is irrational, because only one event will be organized and, even though it will attract the maximum number of participants, it would fail to yield a positive effect $f(S_k)$ on the wellness deeds in general. The choice of higher k was previously shown to be counterproductive (too many events will be offered, but would have only a few participants), yet the sponsor would benefit from issuing fewer rewards. For example, for $k = k_{max}$, an employee with the largest number of preferred k_{max} events might become the only member of the club. This is akin to the median voter scheme (discussed by Barbera et al, 1993). A further consultation in this "white field" is necessary.

3. BARGAINING GAME APPLIED TO BOOLEAN TABLES

Suppose that employees who intend to participate in wellness events have been interviewed in order to reveal their preferences. The resulting data can be arranged in $n \times m$ table $W = \|\alpha_{ij}\|_n^m$, where the entry $\alpha_{ij} = 1$ indicates that an employee i has promised to participate in event j, otherwise $\alpha_{ij} = 0$. In this respect, the primary table W is a collection of Boolean columns, each of which comprises of Boolean elements related to one specific event. In the context of the bargaining game, we can discuss an interaction between the wellness club and the manager. The club choice x is a subset of rows $\langle 1,...,i,...,n \rangle$ denoting the newly recruited club members, whereby a subset y of columns $\langle 1,..., j,...m \rangle$ is the manager's choice – the list of available events. The result of the interaction between the club and the manager can thus represent a sub-table H or a block, denoting the players' joint expectation. The players are designated as Player No. 1 – the club, and Player No. 2 – the manager, and both are driven by the desire to receive the rewards. Let us assume that all employees have approved our three reward regulations.[5] While both players are interested in wellness events, their objectives are different. Player No. 1 might aim to motivate each club member to agree to partake in a greater number of company-sponsored events. Player No. 2, the manager, might desire to subscribe maximum number of participants in each event arranged by the company. Let a pair of utilities (υ, F) denotes the players' No. 1 and No. 2 payoffs while both players will bargain considering all possible expected outcomes (x, y) in the form of sub-tables H of the table W

Our intention in developing a theoretical foundation for our story was to follow the Nash's (1950) axiomatic approach. Unfortunately, as previously observed, some fundamental difficulties arise when adopting similar approach. Below, we summarize each of these difficulties, and propose a suitable equivalent. When proceeding in this direction, we first formulate the Nash's axioms in their original nomenclature before reexamining their essence in our own nomenclature. This approach would allow us to provide the necessary proofs in the sections that follow.

As noted by Nash (1950), "... we may define a two-person expectations as a combination of two one-person expectation. ... A probability combination of two two-person expectations is defined by making the corresponding combinations for their components" (p. 157). Readers are also advised to refer to Sen Axiom 8*1, p. 127, or sets of axioms, as well as Luce and Raiffa (1958), Owen (1968) and von Neumann and Morgenstern (1947), with the latter being particularly relevant for utility index interpretation. Rigorously speaking, the com-

[5] We recall the main regulation that none of the club members, inclusive the moderator, receive their rewards if a certain club member participates in fewer than k events.

pactness and convexity of a feasible set $\boldsymbol{S}$ of utility pairs ensures that any continuous and strictly convex function on $\boldsymbol{S}$ reaches its maximum, while convexity guarantees the maximum point uniqueness.

Let us recall the other Nash axioms. The solution must comply with INV (invariance under the change of scale of utilities); IIA (independence of the irrelevant alternatives); and PAR (Pareto efficiency). Note that, following PAR, the players would object to an outcome s when an outcome s' that would make both of them better off exists. We expect that the players would act from a *strong individual rationality* principle SIR. An arbitrary set $\boldsymbol{S}$ of the utility pairs $s = (s_1, s_2)$ can be the outcome of the game. A disagreement arises at the point $d = (d_1, d_2)$ where both players obtain the lowest utility they can expect to realize – the *status quo* point. A *bargaining problem* is a pair $\langle \boldsymbol{S}, d \rangle$[6] and there exists $s \in \boldsymbol{S}$ such that $s_i > d_i$ for $i = 1,2$ and $d \in \boldsymbol{S}$. A *bargaining solution* is a function $f(\boldsymbol{S}, d)$ that assigns to every bargaining problem $\langle \boldsymbol{S}, d \rangle$ a unique element of $\boldsymbol{S}$. The bargaining solution f satisfies SIR if $f(\boldsymbol{S}, d) > 0$ for every bargaining problem $\langle \boldsymbol{S}, d \rangle$.

The advantage of our approach, which guarantees the same properties, lies in the following. We define a feasible set $\boldsymbol{S}$ of expectations, or in more convenient nomenclature, a feasible set $\boldsymbol{S}$ of alternatives as a collection of table W blocks: $\boldsymbol{S} \subseteq 2^W$. Akin to the disagreement or point of contention in the Nash scheme, we define an empty block $\varnothing$ as a *status quo* option in any set of alternatives $\boldsymbol{S}$, which we call the refusal of choice. Next, given any two alternatives H and H' in $\boldsymbol{S}$, an alternative $H \cup H'$ belongs to $\boldsymbol{S}$. In other words, in our case, the set $\boldsymbol{S}$ of feasible alternatives always forms an upper semi-lattice. If an alternative $H \in \boldsymbol{S}$, it follows that all of its subsets $2^H \subseteq \boldsymbol{S}$. Although these arguments do necessitate further discussion, at this juncture, we will state that this is our equivalent to the convex property and will play the same role in proofs as it does in the Nash scheme.

The Nash theorem asserts that there is a unique bargaining solution $f(\boldsymbol{S}, d)$ for every bargaining problem $\langle \boldsymbol{S}, d \rangle$, which maximizes the product of the players' gains in the set $\boldsymbol{S}$ of utility pairs $(s_1, s_2) \in \boldsymbol{S}$ over the

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6 We use the bold notifications $\boldsymbol{S}$ close to the originals. Notification S is preserved for stable point, see later.

disagreement outcome $d = (d_1, d_2)$. This is a so-called symmetric bargaining solution, which satisfies INV, IIA, PAR, and SYM – players symmetric identify, if and only if

$$f(\boldsymbol{S}, d) = \arg\max_{(d_1,d_2)\leq(s_1,s_2)} (s_1 - d_1)\cdot(s_2 - d_2).$$

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It is difficult to make an *ad hoc* assertion regarding properties that can guarantee the uniqueness of similar solution on Boolean Tables. Nevertheless, in the next section, we claim that our bargaining problem on $\boldsymbol{S} \subseteq 2^W$ has the same symmetric or nonsymmetrical shape:

$$f(\boldsymbol{S}, \varnothing) \equiv f(\boldsymbol{S}) = \arg\max_{H\in\boldsymbol{S}} \upsilon(H)^{\theta} F(H)^{1-\theta}$$

for some $0 \leq \theta \leq 1$ provided that Nash axioms hold.

4. Theoretical Aspects of the Boolean game

Henceforth, the table $W = \left\|\alpha_{ij}\right\|_n^m$ will denote the Boolean table discussed in the preceding section, representing employees' promises to attend wellness events. It is beneficial to examine H rows x, symbolizing the arrival of new members to the club, committed to participating in at least k events. Events form, what we call here, a column's event list y, $k = 2,3,\ldots$, where k represents the reward decision. For each event in the event list y, at least $F(H)$ of club members intend to fulfill their promises. For example, let us consider the number of rows in H pertaining to the gain $\upsilon(H)$ of Player No. 1 (e.g., the club members x common gain $\upsilon(H) = |x|$), while the gain of Player No. 2 (the manager's reward) is represented by $F(H)$.

Let us look at the bargaining problem in conjunction with players' preferences. The expectations of the coming club members $i \in x$ towards the event list y can easily be "raised" by $r_i = \sum_{j\in y} \alpha_{ij}$ if $r_i \geq k$, and $r_i = 0$ if $\sum_{j\in y} \alpha_{ij} < k$, $i \in x$, $j \in y$. Similarly, the manager's expectation towards the same event list y can be "accumulated" by means of table H as $c_j = \sum_{i\in x} \alpha_{ij}$, $j \in y$.

We now consider the Bargaining game scenario on Boolean table in more rigorous mathematical form. Below, we use the notation $H \subseteq W$. The block or sub-table H contained in W will be understood in an ordinary set-theoretical nomenclature, where the Boolean Table W is a set of its Boolean 1-elements. All 0-elements will be dismissed from the consideration. Thus, H as a binary relation is also a subset of W. Henceforth, when referring to an element, we assume that it is a Boolean 1-element.

For an element $\alpha \equiv \alpha_{ij} \in W$ in the row i and column j, we use the similarity index $\pi_{ij} = c_j$, counting only on the Boolean elements belonging to H, $i \in x$ and $j \in y$. As the value of $\pi_{ij} = c_j$ depends on each subset $H \subseteq W$, we may write $\pi_{ij} \equiv \pi \equiv \pi(\alpha, H)$, where the set H represents the π-function parameter. It is evident that our similarity indices π_{ij} may only increase with the "expansion" and decrease with the "shrinking" of the parameter H. This yields the following fundamental definitions:

Definition 1. *Basic monotone property. Monotonic System will be understood as a family* $\{\pi(\alpha, H) : H \in 2^W\}$ *of* π*-functions, such that the set* H *is a parameter with the following monotone property: for two particular values* $L, G \in 2^W$, $L \subset G$ *of the parameter* H, *the inequality* $\pi(\alpha, L) \le \pi(\alpha, G)$ *holds for all elements* $\alpha \in W$. *In ordinary nomenclature, the* π*-function with the definition area* $W \times 2^W$ *is monotone on* W *with regard to the second parameter on* 2^W.

Definition 2. *Using a given arbitrary threshold* u *for a non-empty subset* $H \subseteq W$ *let* $V(H)$ *be the subset* $V(H) = \{\alpha \in W : \pi(\alpha, H) \ge u\}$. *The non-empty* H*-set indicated by* S *is called a stable point with reference to the threshold* u *if* $S = V(S)$ *and there exists an element* $\xi \in S$, *where* $\pi(\xi, S) = u$. *See Mullat (1979, 1981) for a comparable concept. Stable point* $S = V(S)$ *has some important properties, which will be discussed later.*

Definition 3. *By Monotonic System kernel we understand a stable point* $S^* = S_{max}$ *with the maximum possible threshold value* $u^* = u_{max}$.

Libkin et al (1990); Genkin et al (1993); Kempner et al (1997); and Mirkin et al (2002) have investigated similar properties of Monotonic Systems and their kernels. With regard to the current investigation, it is noteworthy to state that, given a Monotonic System in general form, without any reference to any kind of "interpretation mechanism", one can always consider a bargaining game between a coalition H – Player No. 1, with utility function $\upsilon(H)$, and Player No. 2 with the payoff function $F(H) = \min_{\alpha \in H} \pi(\alpha, H)$. Following

Nash theorem, a symmetrical solution has to be found in form (1). We will prove below that our solution has to be found in the symmetrical or non-symmetrical form (2).

Definition 4. *Let* d *be the number of Boolean 1's in table* W. *An ordered sequence* $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{d-1} \rangle$ *of distinct elements in the table* W *is called a defining sequence if there exists a sequence of sets* $W = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *such that:*

A. Let the set $H_k = \{\alpha_k, \alpha_{k+1}, \ldots, \alpha_{d-1}\}$. *The value* $\pi(\alpha_k, H_k)$ *of an arbitrary element* $\alpha_k \in \Gamma_j$, *but* $\alpha_k \notin \Gamma_{j+1}$ *is strictly less than* $F(\Gamma_{j+1})$, $j = 0,1,\ldots,p-1$.

B. *There does not exist in the set* Γ_p *a proper subset* L *that satisfies the strict inequality* $F(\Gamma_p) < F(L)$.

Definition 5. *A defining sequence is complete, if for any two sets* Γ_j *and* Γ_{j+1} *it is impossible to find* Γ' *such that* $\Gamma_j \supset \Gamma' \supset \Gamma_{j+1}$ *while* $F(\Gamma_j) < F(\Gamma') < F(\Gamma_{j+1})$, $j = 0,1,\ldots,p-1$.

It has been established that, in an arbitrary Monotonic System, one can always find a complete defining sequence (see Mullat, 1971, 1976). Each set Γ_j is the largest stable set with reference to the threshold $F(\Gamma_j)$. This allows us to formulate our Frontier Theorem.

Frontier Theorem. *Given a bargaining game on Boolean Tables with an arbitrary set* $\boldsymbol{S}$ *of feasible alternatives* $H \in \boldsymbol{S}$, *the expectations points* $(\upsilon(\Gamma_j), F(\Gamma_j))$, $j = 0,1,\ldots,p$, *of a complete defining sequence* $\overline{\alpha}$ *arrange a Pareto frontier in* $\Re^2$.

Proof. Let $W^S \in \boldsymbol{S}$ be the largest set in $\boldsymbol{S}$ containing all other sets $H \in \boldsymbol{S}$: $H \subseteq W^S$. Let a complete defining sequence $\overline{\alpha}$ [7] exist for W^S. Let the set H^c be the set containing all such sets $V(H)$, where $V(H) = \{\alpha \in W : \pi(\alpha, H) \geq F(H)\}$. Note that $H \subseteq V(H^c)$ and $F(H^c) \geq F(H)$. Now, for accuracy, we must distinguish three situations:

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[7] We are not going to use any new notifications to distinguish between Boolean Tables W and W^S.

(a) In the sequence $\overline{\alpha}$ one can find an index j such that
$F(\Gamma_j) \le F(H^c) < F(\Gamma_{j+1})$ $j = 0,1,\ldots,p-1$;

(b) The case of $F(H^c) < F(W) = F(\Gamma_0)$;

(c) And $F(H) > F(\Gamma_p)$. The case (c) is impossible because, on the set Γ_p, the function $F(H)$ reaches its global maximum.

In case of (b), the expectation $(\upsilon(\Gamma_0), F(\Gamma_0))$, $\Gamma_0 = W$, is more beneficial than $(\upsilon(H), F(H))$, which concludes the proof. In case of (a), let $F(\Gamma_j) < F(H^c)$, otherwise the equality $F(\Gamma_j) = F(H^c)$ is the statement of the theorem (when reading the sentence after the next, the index $j+1$ should be replaced by j). In this case, the set H^c must coincide with Γ_{j+1}, $j = 0,1,\ldots,p-1$, otherwise the defining sequence $\overline{\alpha}$ is incomplete. Indeed, looking at the first element $\alpha_k \in H^c$ in the sequence $\overline{\alpha}$, it can be ascertained that, if $\Gamma_{j+1} = H^c$ does not hold, the set $H_k = H^c$ because it is the largest stable set up to the threshold $F(H^c)$. Hence, the set H_k represents an additional Γ-set in the sequence $\overline{\alpha}$ with the property A of a complete defining sequence. The inequalities $F(\Gamma_{j+1}) = F(H^c) \ge F(H)$, $\upsilon(\Gamma_{j+1}) = \upsilon(H^c) \ge \upsilon(H)$, due to $\Gamma_{j+1} = H^c \supseteq H$ and the basic monotonic property, are true. Thus, the point $(\upsilon(\Gamma_{j+1}), F(\Gamma_{j+1}))$ is more advantageous than $(\upsilon(H), F(H))$. ■

5. Algorithm for Solving the Bargaining Problem

To summarize, the discussion that follows is governed by the Nash bargaining scheme. Some reservations (see, for example, Luce and Raiffa, 6.6) hold as usual because our bargaining game on Boolean Tables is purely atomic, i.e., it does not permit lotteries (which are an important element of any bargaining scenario). Given this restriction, the uniqueness of the Nash solution cannot be immediately guaranteed. It is important to note that "...the Nash solution of $\langle \boldsymbol{S}, d \rangle$ depends only on disagreement point d and the Pareto frontier of $\boldsymbol{S}$. The compactness and convexity of $\boldsymbol{S}$ are important only insofar as they ensure that the Pareto frontier of $\boldsymbol{S}$ is well defined and concave. Rather than starting with the set $\boldsymbol{S}$, we could have imposed our axioms on a problem defined by a non-increasing concave function (and disagreement point d…Osborn and Rubinstein, 1990, p. 24). In our case, $(\upsilon(\Gamma_j), F(\Gamma_j))$, $j = 0,1,\ldots,p$, represents an atomic Pareto frontier. Therefore, it is possible to provide the proof of non-symmetrical solution (see Kalai, 1977, p. 132), as well as perform the derivation with the product of utility gains in its asymmetrical form (2).[8] The problem of maximizing the product is primarily of technical nature. In the discussions that follow, we will introduce an algorithm for that purpose. We will first comment on the individual algorithm step in relation to the definitions.

The algorithm's last iteration, see below, through the step **T** detects the largest kernel $\overline{K} = S^{*}$[9] (Mullat, 1995). The original version (Mullat, 1971) of the algorithm aimed to detect the largest kernel and is akin to a greedy inverse serialization procedure (Edmonds, 1971). The original version of the algorithm produces a complete defining sequence, which is imperative for finding the bargaining solution aligned with the Frontier Theorem. In the context of the current version it fails to produce a complete defining sequence. Rather, it only detects some thresholds u_j, and some stable set $\Gamma_j = S_j$. The sequence $u_0, u_1, \ldots$ is monotonically increasing: $u_0 < u_1 < \ldots$ while the sequence $\Gamma_0, \Gamma_1, \ldots$ is monotonically shrinking: $\Gamma_0 \supset \Gamma_1 \supset \ldots$, whereby the set $\Gamma_0 = W$ is stable towards the threshold $u_0 = F(W) = \min_{(i,j)\in W} \pi_{ij}$. Hence, the original algorithm is always characterized by higher complexity. For finding the bargaining solution, we can still implement an algorithm of lower complexity, which would require modifying the indices $\pi_{ij} = c_j$.

Let us consider the problem of identifying the players' joint choice H_{max} representing a block $\arg\max_{H\in\mathcal{S}} \upsilon(H)^{\theta} F(H)^{1-\theta}$ of the rows x and columns y in the original table W satisfying the property $\sum_{j\in y} \alpha_{ij} \geq k$, $i \in x$.

Let an index $\pi_{ij} = r_i \cdot \upsilon^{\theta} \cdot c_j^{1-\theta}$[10]. The following algorithm solves the problem.

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[8] There are many techniques that guarantee the uniqueness of the product of utility gains. We are not going to discuss this matter here, because this case is rather an exemption than a rule.

[9] It is possible that some smaller kernels exist as well.

[10] This index obeys the basic monotone property as well.

Algorithm.

I. Set the initial values.

1i. Assign the table parameter H to be identical with W, $H \Leftarrow W$. Set the minimum and maximum bounds a, b on the threshold u for $\pi_{ij} \in H$ values.

A. Establish that the next Step **B** produces a non-empty sub-table H. Remember the current status of table H by creating a temporary table H°: $H^\circ \Leftarrow H$.

1a. Test u as $\frac{1}{2} \cdot (a + b)$ using Step **B**. If it succeeds, replace a by u, otherwise replace b by u and H by H°: $H \Leftarrow H^\circ$ – "regret action".

2a. Go to **1a**.

B. Test whether the minimum of $\pi_{ij} \in H$ over i, j can be equal or greater than u.

1b. Delete all rows in H where $r_i = 0$. This Step **B** fails if all rows in H must be deleted, in which case proceed to **2b**. The table H is shrinking.

2b. Delete all elements in columns where $\pi_{ij} \leq u$. This Step **B** fails if all columns in H must be deleted, in which case proceed to **3b**. The table H is shrinking.

3b. Perform Step **T** if no deletions were made in **1b** and **2b**; otherwise go to **1b**.

T. Test whether the global maximum is found. Table H has halted its shrinking.

1t. Among numbers $\pi_{ij} \in H$, find the minimum $\min \leftarrow \pi_{ij}$ and then perform Step **B** with new value $u = \min$. If it succeeds, set $a = \min$ and return to Step **A**; otherwise, terminate the algorithm.

6. Bargaining Game cooperative Aspects Modification

As mentioned earlier, we consider the game of two persons given as the choice of Player Nr.1 as a subset of rows x in the table $W = \left\| \alpha_{ij} \right\|_n^m$ and player Nr.2 as the choice y of columns. Thus, a joint choice (x, y) is made in the form of a sub-table H or block. Below, we everywhere consider this choice as expectation (x, y) in the set-theoretic sense as a subset H of elements of the table W at the intersection of rows x and columns y. The coalition associated with the choice of H in this case is the set of rows. The utility function $\upsilon = \upsilon(H)$ of such a coalition is ambiguous and depends on the player Nr.2 choice y.

A cooperative game is a pair (N, υ), where N symbolizes a set of participants and υ is the game utility function. Function υ is called a super modular if $\upsilon(L) + \upsilon(G) \leq \upsilon(L \cup G) + \upsilon(L \cap G)$ whereas it is sub modular if the inequality sign $\leq$ is replaced by $\geq$, $L, G \in 2^N$. Various properties of supermodular set functions are specified, among others (see Cherenin et al, 1948 and Shapley, 1971). In the appendix, we illustrate a game, which is neither supermodular nor submodular, but rather a mixture of the two, where single and pairwise participants do not receive extra rewards. On the other hand, it is obvious that all properties of supermodular functions υ remain unchanged for submodular $-\upsilon$ utility function or vice versa.

Let the utility function υ of our game when forming a coalition and the manager's choice is represented by some set-theoretic function $\upsilon(H)$. Particularly, it is useful take $\upsilon(H)=|H|$, or some polynomial function p of its argument like $p(|H|)$. The joint marginal contribution to the coalition x of the participant $i \in x$ and, in particular, the marginal expectation of the manager $j \in y$ (the marginal utilities of the participants) can be represented as $\pi(\alpha_{ij},H)=\frac{\partial\upsilon(H)}{\partial i}\cdot\alpha_{ij}\cdot\frac{\partial\upsilon(H)}{\partial j}$ for $\frac{\partial\upsilon(H)}{\partial i}=\upsilon(H+i)-\upsilon(H)$ if $i \notin x$. When participant $i \in x$ leaves the coalition x, then $\frac{\partial\upsilon(H)}{\partial i}=\upsilon(H)-\upsilon(H-i)$. Marginal notation is valid for any $H \in 2^W$, in fact denoted below as $H \cup i \equiv H+i$, and $H \setminus i \equiv H-i$. For manager expectation y, $j \in y$, similar notation $H \pm j$ is used as $\frac{\partial\upsilon(H)}{\partial j}$. We will not distinguish between the situations when the participant $i \in x$ joins a coalition or leaves the coalition x, or the manager count on expectation $j \in y$ or do not count on j when $j \notin y$. We hope that such actions of the participants in our game clearly emphasize the importance of forming a coalition and the manager's choice when a participant is already a member of a coalition or when someone only intends to join the coalition. Exactly the same consideration applies to manager expectations.

Suppose that the interest of a participant i to join the coalition x equals the participant's marginal contribution. A coalition game is convex (concave) if for any pair L and G of coalitions $L \subseteq G \subseteq x$ the inequality $\frac{\partial\upsilon(L)}{\partial i}\le\frac{\partial\upsilon(G)}{\partial i}$ $\left(\frac{\partial\upsilon(L)}{\partial i}\ge\frac{\partial\upsilon(G)}{\partial i}\right)$ holds for each participant $x \in W$. A similar statement can be made regarding the choice of manager $j \in y$.[11]

[11] Shapley (1971) recognized this condition as equivalent, whereby similar marginal in their investigation of some optimization problems (Nemhauser et al, 1978) have been proposed; Muchnik and Shvartser (1987) also pointed to the link between submodular set functions and the Monotonic Systems, see Mullat (1971).

Theorem. *For the bargaining/coalition game to be convex (concave) it is necessary and sufficient for its utility function to be a supermodular (submodular) set function*. Extrapolated (1978) from Nemhauser et al.

Now, in view of the theorem, marginal utilities of participants in the supermodular game motivate them in certain cases to form coalitions. In a modular game, where the utility function is both supermodular and submodular, marginal utilities are indifferent to collective rationality because entering a coalition would not allow anybody to win or lose their respective payments. In contrast, it can be shown that collective rationality is sometimes counterproductive in submodular games. Therefore, in supermodular games, formation of too many coalitions might be unavoidable, resulting in, for example, the grand coalition. In such cases, in Shapley's (1971) words, this leads to a "snowballing" or "band-wagon" effect. On the other hand, submodular games are less cooperative. In order to counteract these "bad motives" of participants in both supermodular and submodular games, we introduce below a second actor – the manager. Hence, we consider a bargaining game between the coalition and the manager.

Convex game induces an accompanied bargaining game with the utility pair $\left(\upsilon(H), F(H)\right)$, where $F(H) = \min_{i \in x} \frac{\partial \upsilon(H)}{\partial i}$; concave game induces utility pair, where $F(H) = \max_{i \in x} \frac{\partial \upsilon(H)}{\partial i}$. Here, the coalition assumes the role of Player No. 1 with the utility function $\upsilon(H)$. The coalition manager, the Player No. 2, expects the reward $F(H)$.

Proposition. *The solution* $f(\boldsymbol{S}, \varnothing)$ *of a Nash's Bargaining Problem* $\langle \boldsymbol{S}, \varnothing \rangle$*, which accompanies a convex (concave) coalition game with utility function* v*, lies on its Pareto frontier* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *maximizing (minimizing) the product* $\upsilon(\Gamma_j)^{\theta} \cdot \frac{\partial \upsilon(\Gamma_j)}{\partial \alpha}^{1-\theta}$ *for some* $j = 0,1,\ldots,p$*, and* $0 \le \theta \le 1$. This statement is a clear corollary from the Frontier Theorem. ■

In accordance with the basic monotonic property, see above, given some monotonic function $\pi(i, H) = \frac{\partial \upsilon(H)}{\partial i}$ on $N \times 2^N$, it is not immediately apparent that there exists some utility function $\upsilon(H)$ for which the function $\pi(i, H)$ constitutes a monotonic marginal utility $\frac{\partial \upsilon(H)}{\partial i}$. The following theorem, accommodated in line with the work of Muchnik and Shvartser (1987), addresses this issue.

The existence Conjecture. *For the function* $\pi(i,H)$ *to represent a monotonic marginal utility* $\frac{\partial \upsilon(H)}{\partial i}$ *of some supermodular (submodular) function* $\upsilon(H)$, *it is necessary and sufficient that*

$$\begin{aligned}&\frac{\partial}{\partial k}\frac{\partial \upsilon(H)}{\partial i} \equiv \pi(i,H) - \pi(i,H-k) = \\ &= \pi(k,H) - \pi(k,H-i) \equiv \frac{\partial}{\partial i}\frac{\partial \upsilon H}{\partial k}\end{aligned} \quad \textit{holds for } i,k \in x \subseteq N.$$

The interpretation of this condition is left for the reader.

7. Discussion

We start this discussion with a heuristic interpretation. Following the apparatus of monotonic systems in terms of data mining (Mullat, 1971), it is reasonable to find the Pareto frontier in terms of the game theory as well. The potential manager's bargaining strategy is presented next. First, in the grand coalition $N \equiv \Gamma_0$, the manager identifies the participants with the least marginal utility $u_0 = F(N) = \min_{i\in N} \frac{\partial \upsilon(N)}{\partial i}$. The manager will advise them to stay in line and wait for their rewards. All participants that have joined the line will be temporarily disregarded in any coalition formation. Following the game convexity, one of the remaining participants (i.e., those still remaining in the coalition formation process) must find themselves worse off owing to the participants in line being excluded from the process. Manager would thus suggest to these participants to also join the line and wait for their rewards. The manager continues the line construction in the same vein. This process will result in a scenario in which all remaining participants Γ_1 (outside the line) are better off than u_0, i.e., better off than those waiting in line for their rewards. Now, the manager repeats the entire procedure upon participants $\Gamma_1, \Gamma_2, \ldots$ until all participants from N are assigned to wait in line to obtain their rewards. Manager, certainly, keeps a record of the events $0,1,\ldots$ and is aware when the marginal utility thresholds increases from u_0 to u_1, etc. It is obvious that the increments are always positive: $u_0 < u_1 < \ldots < u_p$.

What is the outcome of this process? Participants staying in line arrange a nested sequence of coalitions $\left\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \right\rangle$. The most powerful marginal participants, those present when the last event p occurs, form a coalition Γ_p. The next powerful coalition will be Γ_{p-1}, etc., coming back once again to the starting event 0, when the participants arrange the grand coalition $N = \Gamma_0$. Our Frontier theorem guarantees that such a manager bargaining strategy, in convex games, classifies a Pareto frontier $\left\langle (\upsilon(\Gamma_0), u_0), (\upsilon(\Gamma_1), u_1), \ldots, (\upsilon(\Gamma_p), u_p) \right\rangle$ for a bargaining game between the manager and coalitions under formation.[12] Thus, the game ends when a bargaining agreement is reached between the manager and the coalition. However, some participants might still stay in line, waiting in vain for their rewards, because the manager might not agree to allow them to partake in coalition formation. Indeed, due to the existence of those marginal participants, the manager may lose a large portion of his/her reward $F(\Gamma_k)$, for some $k's \in \langle 1, \ldots, p \rangle$.[13]

Only the last issue is relevant to our bargaining solution $\Gamma = f(\boldsymbol{S}, \varnothing)$ to the supermodular bargaining game. The coalition Γ is a stable point with reference to the threshold value $u = F(\Gamma) = \min_{i \in \Gamma} \dfrac{\partial \upsilon(\Gamma)}{\partial i}$. This coalition guarantees a gain $u = F(\Gamma)$ to Player No. 2. Therefore, Player No. 2 can prevent anyone $i \notin \Gamma$ outside the coalition $\Gamma \in \boldsymbol{S}$ from becoming a new participant of the coalition because the outsider's marginal contribution $\dfrac{\partial \upsilon(\Gamma)}{\partial i}$ reduces the gain guaranteed to Player No. 2. The same incentive governing the behavior of Player No. 2 will prevent some members $i \in \Gamma$ from leaving the coalition. The unconventional interpretation given below might help elucidate this situation.

12 This sequence of players/elements in line arranges so-called defining sequence in data mining process.

13 We refer to similar behaviour of players in “Left- and Right-Wing Political Power Design: The Dilemma of Welfare Policy with Low-Income Relief” as political parties bargaining game agents registered under the social security administration.

Let us observe a family of functions on $N \times 2^N$ monotonic towards the second set variable H, $H \in 2^N$. Let it be a function $\pi(i;H) \equiv \frac{\partial \upsilon H}{\partial i}$. We already cited Shapley (1971), who introduced the convex games, with the marginal utility $\frac{\partial \upsilon H}{\partial i} = \upsilon(H) - \upsilon(H-i)$, which is the one of many exact utilizations of the monotonicity $\pi(i,L) \le \pi(i,G)$ for $i \in L \subseteq G$. Authors of some extant studies, including this researcher, refer to these marginal $\upsilon(H) - \upsilon(H-i)$ set functions as the marginal of supermodular functions $\upsilon(H)$. By inverting the inequalities, we obtain submodular set functions.

Convex coalition game, referring to Shapley (1971) once again, can have a "snowballing" or "band-wagon" effect of cooperative rationality; i.e., in a supermodular game, the cooperative rationality suppresses the individual rationality. In contrast, in submodular games with the inverse property $\pi(i,L) \ge \pi(i,G)$ (an extrapolation this time), the individual rationality suppresses the collective rationality. Indeed, according to the rules of the game, the manager's reward will depend on the least marginal utility $u = F(H) = \min_{i \in H} \frac{\partial \upsilon(H)}{\partial i}$ of some of weakest members of the resulting in the coalition H under formation. Indeed, according to the rules of the game, the manager's reward will depend on the lowest marginal utility of some of the weakest members of the resulting coalition. If the utility function is submodular, the positive effect of the health club members' cooperation disappears. Now, we can focus on a two-person game to be played out between the manager and the coalition without consideration of cooperation.

8. Conclusion

To sum up our efforts, all of this was made possible by a category called "Monotonic System", which is a kind of quintessence of the monotonous phenomenon of reality, linking two separate categories: "The problem of bargaining" and "Coalition game" together by one guiding thread. Nash bargaining solution being understood as a point on the Pareto frontier in Monotonic System might be an acceptable convention in the framework of "fast" calculation. The corresponding algorithm for finding the solution is characterized by a relatively few operations and can be implemented using known computer programming "recursive techniques" on tables. From a purely theoretical perspective, we believe that our technique is a valuable addition to the repertoire presently at the disposal of the game theoreticians. Our bargaining solution is presently not fully grounded in validated scientific facts established in game theory. Consultations with specialists in the field are thus necessary to develop our work further. In our view, our coalition formations games are sufficiently clear and do not require specific economic interpretations. Nevertheless, they need to be confirmed by other fundamental studies.

APPENDIX. Neither Super Modular nor Sub Modular Utility Functions Illustration of Bargaining in Club Formation.

Recall the wellness club formation game from Section 2. Given the utility function $\upsilon(x)$, although whether the club members actually arrive at individual payoffs or not is irrelevant, the club formation is still of our interest. Let the game participants $N = \{1,2,3,4,5,6,7\}$ try to organize a club. Let the utility (revenue) function comply with the promises of individual employees to participate in the offered wellness events in accordance with their survey responses, see Table 1. We demand that all five-wellness events be materialized.

Define $$\upsilon(x) = |x| + \sum_{i \in x} \sum_{j=1}^{5} \alpha_{ij}, \text{ where}$$
$$x \subseteq N = \{1,2,3,4,5,6,7\}.$$

In other words, a promise fulfilled by the club member contributes a Bank Note to the participant. In addition to all the promises fulfilled, a side payment per capita is available. According to this rule $\upsilon(\{1\}) = 3$, $\upsilon(\{2\}) = 5$,... Nonetheless, we have changed the side payments rule, so that the game transforms into neither supermodular nor submodular game. Note, that $\sum_{i}^{7} \upsilon(\{i\}) = 22 < \upsilon(N) = \upsilon(\{1,2,3,4,5,6,7\}) = 29$, what renders a non-essential game. If the decision of the CEO becomes k=2, then each member of the wellness club Γ_0, according to the rules of the game, receives one basic banknote, and at the same time, a side payment of 7 banknotes will allow additionally to double the reward if the grand coalition Γ_0 is formed. However, the club manager will not be interested in organizing club Γ_0, since the CEO's reward in organizing clubs Γ_1, Γ_2 and Γ_3 with fewer participants, like on the Pareto frontier, Fig. No. 1-3, according to the rules of the game, allows getting higher rewards.

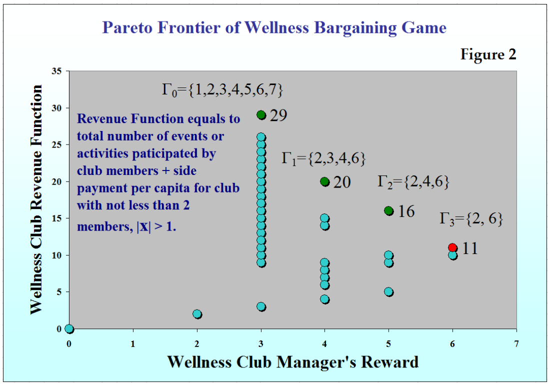
Yes, indeed, the employees, whether they choose to cooperate or not, will be discouraged from forming a club arriving at the same gains. To change the situation into that similar to "*the real life cacophonous*", let the side payment per capita be removed for single and pairwise participants while keeping the rewards intact for all other coalitions for which the size exceeds 2. Thus $\upsilon(\{1\})=2$, $\upsilon(\{2\})=4$, $\upsilon(\{1,2\})=6$, $\upsilon(\{3,6\})=5$, $\upsilon(\{2,3,5\})=12$, etc. The gain, which was defined as $F(x)=\min_{i\in x}\frac{\partial x}{\partial i}\equiv(\upsilon(x)-\upsilon(x-i))$, makes the employees' "cooperative behavior close to grand coalition" less profitable for the manager, see above.

Therefore, the manager would benefit from encouraging the employees to enter the club of a "reasonable size". In Table 8, we examine this phenomenon using different manager gain $F(x)$ values.

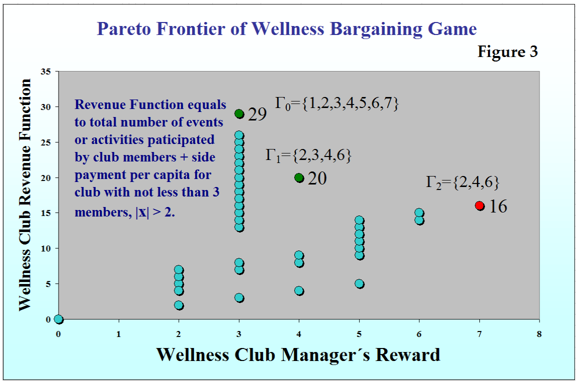
Table 8.

Wellness clubs List							Marginal Utilities p/capita							υ	F
1	2	3	4	5	6	7	1	2	3	4	5	6	7	υ(H)	F(H)
*							2							2	2
	*							4						4	4
*	*						2	4						6	2
		*							3					3	3
*		*					2		3					5	2
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
		*		*					3		2			5	2
*		*		*			5		6		5			10	5
	*	*		*				7	6		5			12	5
*	*	*		*			3	5	4		3			15	3
			*	*						4	2			6	2
*			*	*			5			7	5			11	5
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	.	*	*	*	*	*		.	4	5	3	6	3	21	3
*	.	*	*	*	*	*	3	.	4	5	3	6	3	24	3
.	*	*	*	*	*	*	.	5	4	5	3	6	3	26	3
*	*	*	*	*	*	*	3	5	4	5	3	6	3	29	3

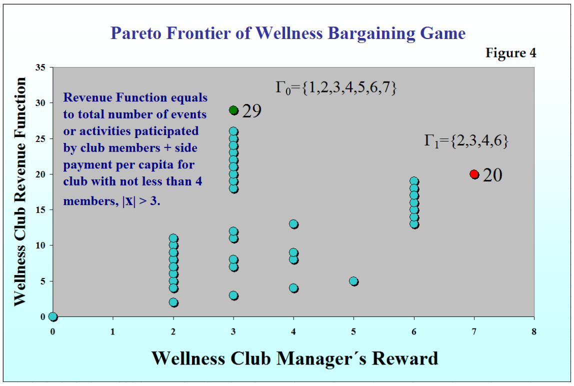
At last, we illustrate the bargaining game in the graph below and make some comments.



N.B. Observe that utility pairs $(29,3)$, $(20,4)$, $(16,5)$ and $(11,6)$ constitute the Pareto frontier of bargaining solutions for the bargaining problem involving the manager as Bargainer No. 1 and coalitions as Bargainer No. 2. Accordingly, given the grand coalition $N = \Gamma_0 = \{1,2,3,4,5,6,7\}$, three proper coalitions $\Gamma_1 = \{2,3,4,6\}$, $\Gamma_2 = \{2,4,6\}$ and $\Gamma_3 = \{2,6\}$ exist. Solutions $\upsilon(\Gamma_1) = 20$, $F(\Gamma_1) = 4$ and $\upsilon(\Gamma_2) = 16$, $F(\Gamma_2) = 5$, maximize the product of participants' gains over the disagreement point $(0,0)$ at $20 \cdot 4 = 16 \cdot 5 = 80$. More specifically, as noted at the beginning of the paper, the solution might not be unique and some external considerations may help. For example, the sponsor expenses for $(20,4)$ are equal to 24, while those pertaining to $(16,5)$ are equal to 21, which might be decisive. That is the case when the bargaining power $\theta = ½$ of the coalitions Γ_1, Γ_2 and the manager are in balance. Otherwise, choosing the coalition bargaining power $\theta < ½$, the manager will be better off materializing the solution $(5,16)$. Conversely, coalition Γ_2 will be better off if $\theta > ½$.



NB. Comparison with Fig. 2 reveals that coalition $\Gamma_3 = \{2,6\}$ is no longer located on the Pareto frontier.



N.B. Comparison with Fig. 3 indicates that coalition $\Gamma_2 = \{2,4,6\}$ no longer lies on the Pareto frontier.

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