

Correspondence between two antimatroid algorithmic characterizations

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Abstract

The basic distinction between already known algorithmic characterizations of matroids and antimatroids is in the fact that for antimatroids the ordering of elements is of great importance.

While antimatroids can also be characterized as set systems, the question whether there is an algorithmic description of antimatroids in terms of sets and set functions was open for some period of time.

This article provides a selective look at classical material on algorithmic characterization of antimatroids, i.e., the ordered version, and a new unordered version. Moreover we empathize formally the correspondence between these two versions.

keywords: *antimatroid, greedoid, chain algorithm, greedy algorithm, monotone linkage function.*

1 Introduction

In this paper we compare two algorithmic characterization of antimatroids. There are many equivalent axiomatizations of antimatroids, that may be separated into two categories: antimatroids defined as set systems and antimatroids defined as languages. Boyd and Faigle [1] introduced an algorithmic characterization of antimatroids based on the second antimatroid definition - as a formal language. Another characterization of antimatroids, that uses their definition as a set systems, is considered in this paper. This approach is based on the optimization of set functions defined as the minimum value of linkages between a set and elements from the set complement. The correspondence between two these approaches is established.

Section 2 gives some basic information about antimatroids as set systems. In Section 3 a set system generated by an isotone operator is introduced, and its equivalence to an antimatroid is proved. In Section 4 monotone linkage functions are considered.

The optimization of the functions defined as the minimum of the monotone linkage functions extends to antimatroids, and the polynomial algorithm that finds an optimal set is constructed. In Section 5 an algorithmic characterization of truncated antimatroids in terms of the monotone linkage functions are considered. In Section 6 the results of Boyd and Faigle are presented and connection between their approach and the approach based on monotone linkage functions is established.

2 Preliminaries

Let E be a finite set. A *set system* over E is a pair (E, \mathcal{F}) where $\mathcal{F} \subseteq 2^E$ is a family of subsets of E , called *feasible* sets. We will use $X \cup x$ for $X \cup \{x\}$ and $X - x$ for $X - \{x\}$.

Definition 2.1 *A non-empty set system (E, \mathcal{F}) is an antimatroid if*

- (A1) *for each non-empty $X \in \mathcal{F}$ there is an $x \in X$ such that $X - x \in \mathcal{F}$*
- (A2) *for all $X, Y \in \mathcal{F}$, and $X \not\subseteq Y$, there exist an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.*

Any set system satisfying (A1) is called *accessible*.

Definition 2.2 *A set system (E, \mathcal{F}) has the interval property without upper bounds if for all $X, Y \in \mathcal{F}$ with $X \subseteq Y$ and for all $x \in E - Y$, $X \cup x \in \mathcal{F}$ implies $Y \cup x \in \mathcal{F}$.*

There are some different antimatroid definitions, for the sake of completeness we will prove the following proposition:

Proposition 2.3 [3][4] *For an accessible set system (E, \mathcal{F}) the following statements are equivalent:*

- (i) *(E, \mathcal{F}) is an antimatroid*
- (ii) *\mathcal{F} is closed under union*
- (iii) *(E, \mathcal{F}) satisfies the interval property without upper bounds*

Proof. (i) \Rightarrow (ii) Let $X, Y \in \mathcal{F}$ and $Y \not\subseteq X$. Repeated application of (A2) yields a set $X \cup (Y - X) \in \mathcal{F}$, i.e. $X \cup Y \in \mathcal{F}$.

(ii) \Rightarrow (iii) Let $X, Y \in \mathcal{F}$ and $X \subseteq Y$ and $x \in E - Y$ and $X \cup x \in \mathcal{F}$, then $(X \cup x) \cup Y \in \mathcal{F}$, i.e. $Y \cup x \in \mathcal{F}$.

(iii) \Rightarrow (i) Let $X, Y \in \mathcal{F}$ and $X \not\subseteq Y$. Accessibility means that we can find a sequence $x_1 x_2 \dots x_k$ and corresponding sequence $\emptyset = X_0 \subset X_1 \subset \dots \subset X_k = X$ where $X_i = X_{i-1} \cup x_i$, and $X_i \in \mathcal{F}$ for $0 \leq i \leq k$. Let j be the least integer for which $X_j \not\subseteq Y$. Then $X_{j-1} \subseteq Y$, $x_j \notin Y$ and $X_{j-1} \cup x_j \in \mathcal{F}$, that implies $Y \cup x_j \in \mathcal{F}$. Hence (E, \mathcal{F}) is an antimatroid. ■

A maximal feasible subset of set $X \subseteq E$ is called a *basis of X* , and will be denoted by \mathcal{B}_X . Clearly, by (ii), there is only one basis for each set.

For a set $X \in \mathcal{F}$, let $\Gamma(X) = \{x \in E - X : X \cup x \in \mathcal{F}\}$ be the set of *feasible continuations* of X [4].

We will say that $\Gamma : \mathcal{F} \rightarrow 2^E$ is an *isotone operator* if for all $X, Y \in \mathcal{F}$, $X \subseteq Y$ implies $\Gamma(X) \cap (E - Y) \subseteq \Gamma(Y)$. An accessible set system (E, \mathcal{F}) satisfies the interval property without upper bounds if and only if $\Gamma : \mathcal{F} \rightarrow 2^E$ is an isotone operator:

$$x \in \Gamma(X) \cap (E - Y) \Leftrightarrow (x \in E - Y) \wedge (X \cup x \in \mathcal{F}) \Rightarrow Y \cup x \in \mathcal{F} \Leftrightarrow x \in \Gamma(Y).$$

3 Isotone operators and antimatroids

In this section a characterization of an antimatroid as a set system generating by an isotone operator is given.

Consider an operator $\Psi : 2^E \rightarrow 2^E$ such that

$$\text{for each } X \subset E, \Psi(X) \subseteq E - X \tag{1}$$

In what follows we will use only operators satisfied (1). We can build a set system, denoted by $(E, \mathcal{F}(\Psi))$, using the following algorithm.

Ψ - Algorithm

1. $\mathcal{F}(\Psi) = \emptyset$
2. $i = 0$
3. $T_i = \{\emptyset\}$
4. while $T_i \neq \emptyset$
 - 4.1 $\mathcal{F}(\Psi) = \mathcal{F}(\Psi) \cup T_i$
 - 4.2 $T_{i+1} = \{X \cup x : (X \in T_i) \wedge (x \in \Psi(X))\}$
 - 4.3 $i = i + 1$

Clearly, $(E, \mathcal{F}(\Psi))$ is an accessible system, because for each non empty $X \in \mathcal{F}(\Psi)$ there exists its "parent" $X - x \in \mathcal{F}(\Psi)$ by using which the set X was generated.

It is not difficult to see that the property (1) implies that on each step i the algorithm generates the set T_i in which each set has exactly i elements.

Here are some examples of Ψ operators:

(a) Let for each $X \subset E$, $\Psi(X) = E - X$. Then $\mathcal{F}(\Psi) = 2^E$.

(b) Let $E = \{x_1, x_2, \dots, x_n\}$, where for each i , $x_i < x_{i+1}$. Define for each $\emptyset \subset X \subset E$, $\Psi(X) = \{x \in E : x > \max(X)\}$, and $\Psi(\emptyset) = E$. It is easy to see that, $\mathcal{F}(\Psi) = 2^E$.

(c) Let (E, \leq) is a poset and $\Psi(X) = \{x \in E : x = \min(E - X)\}$, then the obtained set system $(E, \mathcal{F}(\Psi))$ is a poset antimatroid [4].

(d) Let $E = \{x_1, x_2, \dots, x_n\}$, where for each i , $x_i < x_{i+1}$. Define for each $\emptyset \subset X \subset E$, $\Psi(X) = \{x_{i+1}\}$, where $x_i = \max(X)$, and $\Psi(\emptyset) = \{x_1\}$. Then $\mathcal{F}(\Psi)$ is a chain $\emptyset \subset \{x_1\} \subset \{x_1, x_2\} \subset \dots \subset \{x_1, x_2, \dots, x_n\}$.

Note, that the same set systems may be generated by different operators (see the above examples). Now assume, that the operator Ψ is also an isotone operator, i.e.,

$$\text{if } X, Y \subset E \text{ then } X \subseteq Y \text{ implies } \Psi(X) \cap (E - Y) \subseteq \Psi(Y). \tag{2}$$

Lemma 3.1 *Let Ψ be an isotone operator, then for each $X \in \mathcal{F}(\Psi)$ and $x \in E - X$, $X \cup x \in \mathcal{F}(\Psi)$ if and only if $x \in \Psi(X)$.*

Proof. "If" immediately follows from the structure of the Ψ -Algorithm.

Conversely, if set $X \cup x$ was generated from X then $x \in \Psi(X)$. If not - there is a sequence of sets generated by the Ψ -Algorithm $\emptyset = X_0 \subset X_1 \subset \dots \subset X_k = X \cup x$ such that $X_i = X_{i-1} \cup x_i$ where $x_i \in \Psi(X_{i-1})$. Let $x = x_j$. Then

$$(X_{j-1} \subseteq X) \wedge (x \in \Psi(X_{j-1})) \wedge (x \notin X) \text{ that implies (from (2)) } x \in \Psi(X). \blacksquare$$

Corollary 3.2 *Two isotone operators Ψ_1 and Ψ_2 generate the same set system (E, \mathcal{F}) if and only if $\Psi_1|_{\mathcal{F}} = \Psi_2|_{\mathcal{F}}$.*

The property (2) makes possible to generate an antimatroid.

Theorem 3.3 *Set system (E, \mathcal{F}) is an antimatroid if and only if there exists an isotone operator Ψ such that $\mathcal{F} = \mathcal{F}(\Psi)$.*

Proof. Each $(E, \mathcal{F}(\Psi))$ is an accessible system. If in addition Ψ is an isotone operator, then the set system $(E, \mathcal{F}(\Psi))$ satisfies the interval property without upper bounds. Indeed, if $X, Y \in \mathcal{F}(\Psi)$, and $X \subseteq Y$ and $x \in E - Y$, and $X \cup x \in \mathcal{F}(\Psi)$, then (from Lemma 3.1) $x \in \Psi(X)$, that implies $x \in \Psi(Y)$, i.e. $Y \cup x \in \mathcal{F}(\Psi)$. It means (see Proposition 2.3) that the set system $(E, \mathcal{F}(\Psi))$ is an antimatroid. Moreover, $\Psi(X) = \Gamma(X)$ for each $X \in \mathcal{F}(\Psi)$.

Conversely, let (E, \mathcal{F}) be an antimatroid. We will show that this antimatroid can be generated by some isotone operator. First, build the operator $\Psi : 2^E \rightarrow 2^E$:

$$\text{for each } X \subset E, \Psi(X) = \Gamma(\mathcal{B}_X) \tag{3}$$

where \mathcal{B}_X is a basis of X . Since the basis is unique, the definition is correct.

To show that the constructed operator is a required isotone operator we have to prove two properties:

$$(i) \Psi(X) \subseteq E - X$$

Indeed, suppose that $x \in \Psi(X)$ and $x \in X$, then $\mathcal{B}_X \cup x \subseteq X$ and $\mathcal{B}_X \cup x \in \mathcal{F}$. Thus $\mathcal{B}_X \cup x$ is also a basis, a contradiction.

$$(ii) \Psi \text{ satisfies (2).}$$

At first, if $X \subseteq Y$ then $\mathcal{B}_X \subseteq \mathcal{B}_Y$. Now, since Γ is an isotone operator we have, by using (3),

$$(x \in E - Y) \wedge (x \in \Psi(X)) \Rightarrow (x \in E - \mathcal{B}_Y) \wedge (x \in \Gamma(\mathcal{B}_X)) \Rightarrow x \in \Gamma(\mathcal{B}_Y) \Rightarrow x \in \Psi(Y)$$

It remains to show that $\mathcal{F}(\Psi) = \mathcal{F}$. For this purpose, consider $X \in \mathcal{F}$. There exists a sequence $\emptyset = X_0 \subset X_1 \subset \dots \subset X_k = X$ where $X_i = X_{i-1} \cup x_i$ and $X_i \in \mathcal{F}$ for $0 \leq i \leq k$. Thus, $x_i \in \Psi(X_{i-1})$, i.e. elements of the sequence is also obtained by Ψ -generator, and so $X \in \mathcal{F}(\Psi)$.

Conversely, let $X \in \mathcal{F}(\Psi)$. Then there is a sequence $\emptyset = X_0 \subset X_1 \subset \dots \subset X_m = X$ where $X_i = X_{i-1} \cup x_i$ and $x_i \in \Psi(X_{i-1})$. Then $X_i \in \mathcal{F}$ for $0 \leq i \leq m$, and so $X \in \mathcal{F}$. \blacksquare

4 The Chain Algorithm and monotone linkage functions

In general, to optimize a set function is an *NP*-hard problem, but for some specific functions and for some specific set systems polynomial algorithms are known. In this section we consider set functions defined as minimum values of monotone linkage functions. Such set functions can be maximized by a greedy type algorithm over a family of all subsets of E (see [8]). Here we extend this result to antimatroids.

The monotone linkage functions were introduced by Mulla [7]. We will give a necessary basic notions.

Let $\pi : E \times 2^E \rightarrow \mathbf{R}$ be a monotone linkage function such that

$$\text{if } X, Y \subseteq E \text{ and } x \in E, \text{ then } X \subseteq Y \text{ implies } \pi(x, X) \geq \pi(x, Y) \quad (4)$$

For example, the single linkage $\pi(x, X) = \min_{y \in X} d_{xy}$, where d_{xy} is a distance between two objects, is a monotone linkage function.

Consider $F : 2^E \rightarrow \mathbf{R}$ defined for each $X \subset E$

$$F(X) = \min_{x \in E-X} \pi(x, X) \quad (5)$$

These functions were studied in [8],[5]. A simple polynomial algorithm which finds a set $X \subset E$ such that

$$F(X) = \max\{F(Y) : Y \subset E\}$$

was developed, and the idea of this algorithm was used in searching of a protein sequence alignment [6]. In this section we extend our results to a set system $(E, \mathcal{F}(\Psi))$ generated by an isotone operator Ψ . For this purpose we define a new set function as follows:

$$F_{\Psi}(X) = \min_{x \in \Psi(X)} \pi(x, X) \quad (6)$$

It should be pointed out that the definition (6) is not limited to set systems $(E, \mathcal{F}(\Psi))$, but in order to the function F_{Ψ} to be defined on each subset $X \subset E$ the operator Ψ must be non-empty for each subset of E , i.e.,

$$\text{for each } X \subset E, \Psi(X) \neq \emptyset. \quad (7)$$

It is easy to show that a set system $(E, \mathcal{F}(\Psi))$ with non-empty isotone operator Ψ is an antimatroid in which $E \in \mathcal{F}(\Psi)$. In [4] this is a necessary condition for antimatroids, whereas other authors doesn't involve this property in the definition of an antimatroid. Thus, [2] sets these antimatroids to the special class of *normal* antimatroids, and [3] calls them *full* antimatroids. In any case, an antimatroid (E, \mathcal{F}) has one and only one maximal feasible set, namely $\cup_{X \in \mathcal{F}} X$, that we denote as $E_{\mathcal{F}}$. Further we will only need the assumption that operator Ψ is not-empty on $\mathcal{F} - E_{\mathcal{F}}$.

Consider the following optimization problem - given a monotone linkage function π , and a set system $(E, \mathcal{F}(\Psi))$ generated by an isotone operator Ψ , find the feasible set $X \in \mathcal{F}(\Psi) - E_{\mathcal{F}(\Psi)}$ such that $F_{\Psi}(X) = \max\{F_{\Psi}(Y) : Y \in \mathcal{F}(\Psi) - E_{\mathcal{F}(\Psi)}\}$, where

function F_Ψ defined by (6). To solve this problem we build the following algorithm.

The Chain Algorithm (E, π, Ψ)

1. Set $X^0 = \emptyset$
2. Set $X = \emptyset$
3. While $\Psi(X) \neq \emptyset$ do
 - 3.1 If $F_\Psi(X) > F_\Psi(X^0)$, set $X^0 = X$
 - 3.2 Choose $x \in \Psi(X)$ such that $\pi(x, X) \leq \pi(y, X)$ for all $y \in \Psi(X)$
 - 3.3 Set $X = X \cup x$
4. Return X^0

Thus, the Chain Algorithm generates the chain of sets

$$\emptyset = X_0 \subset X_1 \subset \dots \subset X_k = E_{\mathcal{F}(\Psi)},$$

The Chain Algorithm is a greedy type algorithm since it based on the best choice principle: it chooses on each step the extreme element (in sense of linkage function) and thus approaches the optimal solution. Let P is the maximum complexity of $\pi(x, X)$ computation over all pairs (x, X) where $x \in E - X$. Then the Chain Algorithm finds the optimal feasible set in $O(P|E|^2)$ time, for example, in some clustering problems [5], the complexity of the Chain Algorithm is $O(|E|^3)$.

Notice, that for antimatroids the functions Ψ and Γ are identical (see Theorem 3.3) then the following central result immediately follows from previous theorems:

Theorem 4.2 *For an accessible set system (E, \mathcal{F}) , where $\Gamma(X) \neq \emptyset$ for each $X \in \mathcal{F} - E_{\mathcal{F}}$ the following statements are equivalent*

- (1) *the set system (E, \mathcal{F}) is an antimatroid*
- (2) *The Chain Algorithm finds a feasible set that maximizes the function F_{Γ} for every monotone linkage function π*

5 Truncated antimatroids

In this section we extend our results obtained in Section 4 to *truncated antimatroids* considered in the work of Boyd and Faigle [1].

Definition 5.1 *The k -truncation of a set system (E, \mathcal{F}) is a set system defined by*

$$\mathcal{F}_k = \{X \in \mathcal{F} : |X| \leq k\}$$

If \mathcal{F} is an antimatroid, then \mathcal{F}_k is a k -truncated antimatroid.

The *rank* of a set $X \subseteq E$ is defined as $\rho(X) = \max\{|Y| : (Y \in \mathcal{F}) \wedge (Y \subseteq X)\}$, the rank of set system (E, \mathcal{F}) is defined as $\rho(\mathcal{F}) = \rho(E)$. For a given antimatroid (E, \mathcal{F}) the rank of k -truncated antimatroid $\rho(\mathcal{F}_k) = k$, whenever $k \leq \rho(\mathcal{F})$.

Let $(E, \mathcal{F}(\Psi))$ be an antimatroid generated by an isotone operator Ψ . Consider a $(k-1)$ -truncated operator

$$\Psi_{k-1}(X) = \begin{cases} \Psi(X), & |X| \leq k-1 \\ \emptyset, & \text{otherwise} \end{cases} \quad (8)$$

The set system generated by Ψ_{k-1} -operator is a k -truncated antimatroid $(E, (\mathcal{F}(\Psi))_k)$, i.e., $(\mathcal{F}(\Psi))_k = \mathcal{F}(\Psi_{k-1})$. Indeed, any set $|Y| \leq k$ is generated by Ψ_{k-1} -generator if and only if $Y \in \mathcal{F}$, since $\Psi_{k-1}(X) \equiv \Psi(X)$ for all $|X| \leq k-1$. Moreover, assume a set Y for which $|Y| > k$ was obtained by Ψ_{k-1} -generator, then there is a set X such that $|X| \geq k$ and $\Psi_{k-1}(X) \neq \emptyset$, in contradiction with the definition (8).

Clearly, that the Ψ_{k-1} -operator is not isotone on all $2^E - E$, but it satisfies to the following condition:

$$X, Y \subset E, (X \subseteq Y) \wedge (|Y| \leq k-1) \text{ implies } \Psi_{k-1}(X) \cap (E - Y) \subseteq \Psi_{k-1}(Y) \quad (9)$$

We call an operator Ψ a $(k-1)$ -isotone operator if it satisfies (9) and $\Psi(X) = \emptyset$ for each $X \subset E$, such that $|X| \geq k$.

The following theorem shows that a $(k-1)$ -isotone operator generates a truncated antimatroids in the same way as an isotone operator determines an antimatroid:

Theorem 5.2 *Set system (E, \mathcal{F}) is a k -truncated antimatroid if and only if there exists a $(k-1)$ -isotone operator Ψ such that $\mathcal{F} = \mathcal{F}(\Psi)$.*

Proof. To prove that the set system $(E, \mathcal{F}(\Psi))$ is a k -truncated antimatroid we have to build an antimatroid of which it is a truncation. Define, by analogy with [1]

$$\Omega = \{X \subseteq E : \text{there are some } X_1, \dots, X_p \in \mathcal{F}(\Psi) \text{ such that } X = X_1 \cup \dots \cup X_p\}, \quad (10)$$

i.e., Ω is a closure by union of $\mathcal{F}(\Psi)$.

The set system (E, Ω) is closed under union, so to prove that it is an antimatroid it is remain to verify that the set system (E, Ω) is accessible. By analogy with [1] consider a set $X \in \Omega$ and let $X = X_1 \cup \dots \cup X_k$. Then there exists $x \in X_1$ such that $X_1 - x \in \mathcal{F}(\Psi)$. Assume without loss of generality that $x \notin X_2, X_3, \dots, X_k$, for otherwise we could let $X_1 = X_1 - x$. If so, $X - x = (X_1 - x) \cup X_2 \cup \dots \cup X_k \in \Omega$.

To show that the k -truncation of (E, Ω) is $(E, \mathcal{F}(\Psi))$ it is sufficient to prove that $X \in \mathcal{F}(\Psi)$ if and only if $X \in \Omega$ and $|X| \leq k$. Indeed, if $X \in \mathcal{F}(\Psi)$ then, from (10), $X \in \Omega$ and obviously $|X| \leq k$. Conversely, let $X \in \Omega$ and $|X| \leq k$, then $X = A_1 \cup \dots \cup A_p$. We show that $X \in \mathcal{F}(\Psi)$ by induction on p . If $p = 1$ then $X \in \mathcal{F}(\Psi)$. Consider $A = A_1 \cup \dots \cup A_{p-1}$. By the hypothesis of induction, $A \in \mathcal{F}(\Psi)$. Assume $|A| < k$, for otherwise $X = A$ and then $X \in \mathcal{F}(\Psi)$. Let $\emptyset = X_0 \subset X_1 \subset \dots \subset X_l = A$ be a sequence of sets generated by the Ψ -operator, where $X_i = X_{i-1} \cup x_i$ and $x_i \in \Psi(X_{i-1})$ for $1 \leq i \leq l < k$. Let j be the least integer for which $X_j \not\subseteq A_p$. Then $X_{j-1} \subseteq A_p$, $x_j \notin A_p$ and $X_{j-1} \cup x_j \in \mathcal{F}(\Psi)$, that implies $x_j \in \Psi(A_p)$. Repeated application of (9) yields a set $X = A_p \cup (A - A_p) \in \mathcal{F}(\Psi)$.

Conversely, let (E, \mathcal{F}) be a k -truncated antimatroid, then there is an antimatroid (E, Φ) for which (E, \mathcal{F}) is a k -truncation. Since (E, Φ) is an antimatroid, there exists (Theorem 3.3) an isotone operator $\widehat{\Psi}$ that generates the antimatroid (E, Φ) , and $(k-1)$ -truncation $\widehat{\Psi}_{k-1}$ generates the set system (E, \mathcal{F}) . Obviously, the operator $\widehat{\Psi}_{k-1}$ satisfies to (9). ■

Now, using the same technique as in Theorem 4.1 we obtain an algorithmic characterization of truncated antimatroids.

Theorem 5.3 *For an accessible set system (E, \mathcal{F}) , where $\Gamma(X) \neq \emptyset$ for each $X \in \mathcal{F}$ such that $|X| < k$ the following statements are equivalent*

- (1) *the set system $(E, \mathcal{F})_k$ is a k -truncated antimatroid*
- (2) *the Chain Algorithm finds a feasible set that maximizes the function F_Γ on $(E, \mathcal{F})_{k-1}$ for any monotone linkage function π*

6 Correspondence between two algorithmic characterization of antimatroids

In this section we consider another algorithmic approach to antimatroids introduced in work of Boyd and Faigle [1]. Since the work based on other definition of an antimatroid as a formal language, some additional notation is needed. Given a finite alphabet E consists of *letters*. A *word* over E is a sequence of letters from E , denoted by the

lower case of Greek letters α, β and γ . A *language* \mathcal{L} is a set of words of E . The concatenation of two words α and β will be denoted $\alpha\beta$, α_k will be used to denote a word of length k and the set of distinct letters in a word α will be denoted $\tilde{\alpha}$. The language is called *simple* if there are no words with repeated letters.

Definition 6.1 *An antimatroid language is a simple language (E, \mathcal{L}) satisfying the following two properties:*

- (1) *If $\alpha x \in \mathcal{L}$, then $\alpha \in \mathcal{L}$.*
- (2) *If $\alpha, \beta \in \mathcal{L}$ and $\tilde{\alpha} \not\subseteq \tilde{\beta}$, then there exists an $x \in \tilde{\alpha}$ such that $\beta x \in \mathcal{L}$.*

Antimatroids and antimatroid languages are equivalent in the following sense [4].

Theorem 6.2 *If (E, \mathcal{L}) is an antimatroid language, then*

$$F(\mathcal{L}) = \{\tilde{\alpha} : \alpha \in \mathcal{L}\}$$

is an antimatroid $(E, F(\mathcal{L}))$.

Conversely, if (E, \mathcal{F}) is an antimatroid, then

$$L(\mathcal{F}) = \{x_1 \dots x_k : \{x_1, \dots, x_j\} \in \mathcal{F} \text{ for } 1 \leq j \leq k\}$$

is an antimatroid language $(E, L(\mathcal{F}))$. Further, $L(F(\mathcal{L})) = \mathcal{L}$ and $F(L(\mathcal{F})) = \mathcal{F}$.

The next problem was considered in [1]: let $f : E \times 2^E \rightarrow \mathbf{R}$ be a monotone function such that $f(x, A) \leq f(x, B)$ whenever $B \subseteq A$. Define a *maximum nesting function*

$$W(x_1 \dots x_k) = \max\{f(x_1, \{x_1\}), \dots, f(x_k, \{x_1, \dots, x_k\})\}.$$

The *minimax nesting problem* was defined as follows: given a simple language (E, \mathcal{L}) with a monotone function f and a nonnegative integer $k \leq \varrho(\mathcal{L})$, find $\alpha_k \in \mathcal{L}$ such that

$$W(\alpha_k) = \min\{W(\beta_k) : \beta_k \in \mathcal{L}\}$$

The main theorem proved in [1] is reads as follows.

Theorem 6.3 *Let (E, \mathcal{L}) be a simple language. The greedy algorithm solves the minimax nesting problem for every monotone function f if and only if (E, \mathcal{L}) is a truncated antimatroid.*

We will show the correspondence between our algorithmic characterization of antimatroids and characterization of Boyd and Faigle.

First note, that in [1] was proved that the constructed word $\alpha_k = x_1 \dots x_k$ satisfies also the following property:

$$W(x_1 \dots x_i) = \min\{W(\beta_i) : \beta_i \in \mathcal{L}\} \text{ for each } i \text{ such that } 1 \leq i \leq k \quad (11)$$

Second note, that the Chain Algorithm builds a sequence $\emptyset = X_0 \subset X_1 \subset \dots \subset X_k$ where $X_i = X_{i-1} \cup x_i$ for $1 \leq i \leq k$, i.e. this algorithm generates the sequence $x_1 \dots x_k$. So all sets X_i , obtained by the Chain Algorithm, has a natural order: $X_i = \{x_1, \dots, x_i\}$, i.e. we can consider each set X_i also as a word $\alpha_i = x_1 \dots x_i$. We are now ready to prove:

Theorem 6.4 Let (E, \mathcal{L}) be a k -truncated antimatroid and Ψ is a operator which generates this antimatroid, let

$$f(x_i, \{x_1, \dots, x_i\}) = \pi(x_i, \{x_1, \dots, x_{i-1}\}) \text{ for each } i \text{ such that } 1 \leq i \leq k$$

then

(i) if X^0 is an optimal set obtained by the Chain Algorithm, then there exists a word $\alpha_k \in \mathcal{L}$ that satisfies (11) and $X^0 = \{x_1, \dots, x_p\}$ is a shortest prefix of α_k such that $W(x_1 \dots x_{p+1}) = W(\alpha_k) = F_\Psi(X^0)$.

(ii) if α_k is a solution of the minimax nesting problem obtained by the greedy algorithm, then a shortest prefix $\{x_1, \dots, x_p\}$ of α_k such that $W(x_1 \dots x_{p+1}) = W(\alpha_k)$ maximizes the function F_Ψ .

Proof. (i) Let $x_1 \dots x_k$ be the sequence generating by the Chain Algorithm and let $X^0 = \{x_1, \dots, x_p\}$. Set $\alpha_k = x_1 \dots x_k$ and prove that α_k satisfies (11). Suppose not, then let $\gamma_m = y_1 \dots y_m$ be a shortest word such that $W(\gamma_m) < W(x_1 \dots x_m)$. It means that for each $i < m$

$$\max\{\pi(x_1, \emptyset), \dots, \pi(x_i, \{x_1, \dots, x_{i-1}\})\} \leq \max\{\pi(y_1, \emptyset), \dots, \pi(y_i, \{y_1, \dots, y_{i-1}\})\}$$

and for each $i \leq m$

$$\pi(x_m, \{x_1, \dots, x_{m-1}\}) > \max\{\pi(y_1, \emptyset), \dots, \pi(y_i, \{y_1, \dots, y_{i-1}\})\} \quad (12)$$

If $\{y_1 \dots y_{m-1}\} = \{x_1, \dots, x_{m-1}\}$, then $y_m \in \Psi(\{x_1, \dots, x_{m-1}\})$ and from (12)

$$\pi(y_m, \{x_1, \dots, x_{m-1}\}) = \pi(y_m, \{y_1, \dots, y_{m-1}\}) < \pi(x_m, \{x_1, \dots, x_{m-1}\})$$

So the Chain Algorithm should choose y_m and not x_m .

Thus, let j be the smallest index such that $\{y_1, \dots, y_{j-1}\} \subseteq \{x_1, \dots, x_{m-1}\}$ and $y_j \notin \{x_1, \dots, x_{m-1}\}$. Since $y_j \in \Psi(\{y_1, \dots, y_{j-1}\})$, we get that $y_j \in \Psi(\{x_1, \dots, x_{m-1}\})$. Hence, from monotonicity of π and from (12)

$$\pi(y_j, \{x_1, \dots, x_{m-1}\}) \leq \pi(y_j, \{y_1, \dots, y_{j-1}\}) < \pi(x_m, \{x_1, \dots, x_{m-1}\})$$

contradiction to optimal choice of x_m .

Finally, the Chain Algorithm construction implies, that $X^0 = \{x_1, \dots, x_p\}$ is the shortest prefix of α_k such that

$$F_\Psi(X^0) = \pi(x_{p+1}, \{x_1 \dots x_p\}) = W(x_1 \dots x_{p+1}) = W(\alpha_k)$$

(ii) Conversely, let α_k be a solution of the minimax nesting problem and let $X^0 = \{x_1, \dots, x_p\}$ be the shortest prefix such that $W(x_1 \dots x_{p+1}) = W(\alpha_k)$. Then

$$\pi(x_{p+1}, \{x_1 \dots x_p\}) > \pi(x_{i+1}, \{x_1 \dots x_i\}) \text{ for } i < p$$

and

$$\pi(x_{p+1}, \{x_1 \dots x_p\}) \geq \pi(x_{i+1}, \{x_1 \dots x_i\}) \text{ for } i \geq p$$

Certainly, $\pi(x_{p+1}, \{x_1 \dots x_p\}) = \min_{x \in \Psi(X^0)} \pi(x, \{x_1 \dots x_p\})$. If not, there is $x^0 \in \Psi(X^0)$ such that $\pi(x^0, \{x_1 \dots x_p\}) < \pi(x_{p+1}, \{x_1 \dots x_p\})$, i.e. $W(x_1 \dots x_p x^0) < W(x_1 \dots x_p x_{p+1})$ - contradiction to (11). So, $F_\Psi(X^0) = \pi(x_{p+1}, \{x_1 \dots x_p\})$.

Consider some set $X \in F(\mathcal{L})$. If $X = \{x_1 \dots x_j\}$, i.e. X is some prefix of α_k , then

$$F_\Psi(X) = \min_{x \in \Psi(X)} \pi(x, X) \leq \pi(x_{j+1}, \{x_1 \dots x_j\}) \leq \pi(x_{p+1}, \{x_1 \dots x_p\}) = F_\Psi(X^0)$$

Otherwise, let j be the smallest index such that $\{x_1 \dots x_j\} \subseteq X$ and $x_{j+1} \notin X$. Then $x_{j+1} \in \Psi(X)$. Hence,

$$\begin{aligned} F_\Psi(X) &= \min_{x \in \Psi(X)} \pi(x, X) \leq \pi(x_{j+1}, X) \leq \\ &\leq \pi(x_{j+1}, \{x_1 \dots x_j\}) \leq \pi(x_{p+1}, \{x_1 \dots x_p\}) = F_\Psi(X^0). \end{aligned}$$

■

7 Conclusions

In this article, we discussed a set system algorithmic description of one subclass of greedoids, namely, antimatroids. Further we compared a new description with a known one based on the approach to define greedoids as languages. Actually, there are some more important subclasses of greedoids also enjoying natural algorithmic characterizations in terms of their feasible set systems, for instance, matroids and Gaussian greedoids. These results may lead to new algorithmic frameworks for additional types of greedoids. We consider the family of interval greedoids as a strong candidate for the collection of successes of the set system algorithmic approach.

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